1. (5 points) Textbook by Trefethen and Bau, Exercise 27.3.

2. (5 points) Textbook by Trefethen and Bau, Exercise 27.5.

3. (5 points) Textbook by Trefethen and Bau, Exercise 35.6.

4. (Computer Problem, 20 points)

Consider the integral equation
\[ \phi(x) - \int_a^b K(x, y) \phi(y) \, dy = f(x), \quad a \leq x \leq b, \]  
(1)
where \( K(x, y) \) and \( f(x) \) are given and the function \( \phi(x) \) is the unknown. A numerical method for solving the integral equation (1), Nystrom’s method, requires the integral equation is satisfied at \( n \) discrete points, \( \{x_j, j = 0, 1, \cdots, n\} \) on the interval \([a, b]\) and approximates the integral with numerical quadratures (such as composite trapezoidal rule). Thus, by Nystrom’s method, the integral equation is discretized into a linear system
\[ \phi_j - \sum_{k=0}^{n} \alpha_k K(x_j, x_k) \phi_k = f(x_j), \quad j = 0, 1, \cdots, n, \]  
(2)
where \( \phi_j \) is the numerical approximation (solution) of \( \phi(x_j) \) and \( \alpha_k \) are the coefficients/weights of the numerical integration scheme (the quadrature).

An example of the integral equation of the form (1) is
\[ \phi(x) - \frac{1}{2} \int_0^1 (x + 1)e^{-xy} \phi(y) \, dy = e^{-x} - \frac{1}{2} + \frac{1}{2} e^{-(x+1)}, \quad 0 \leq x \leq 1. \]  
(3)
The analytic solution to (3) is \( \phi(x) = e^{-x} \).

(a) Solve the integral equation (3) by Nystrom’s method with composite trapezoidal rule as follows. Let \( x_j = jh, j = 0, 1, \cdots, n \) evenly divide the interval \([0, 1]\) with \( h = 1/n \). What are \( \alpha_k \) if we use composite trapezoidal rule? Solve the resulting linear systems by Gaussian elimination with partial pivoting for \( n = 64, 256, 1024, 4096 \) and find the error of the numerical solution at \( x = 1 \) for each \( n \). Find how much computational time is needed for solving the linear system corresponding to \( n = 4096 \).

(b) Repeat the above exercises using GMRES for solving the linear systems. What’s the computational time in this case for \( n = 4096 \)? Choose the residual error tolerance as large as possible as long as the error of the numerical solution is not affected by the tolerance.