1. Exercise 4.2 of the textbook by Iserles.
Consider the solution of $y' = \Lambda y$ where

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{C}^-. $$

a. Prove that

$$\Lambda^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}, \quad n = 0, 1, \ldots.$$

b. Let $g$ be an arbitrary function that is analytic about the origin. The $2 \times 2$ matrix $g(\Lambda)$ can be defined by substituting powers of $\Lambda$ into the Taylor expansion of $g$. Prove that

$$g(t\Lambda) = \begin{pmatrix} g(t\lambda) & tg^{(1)} \lambda \\ 0 & g(t\lambda) \end{pmatrix}. $$

c. By letting $g(z) = e^z$ prove that $\lim_{t \to \infty} y(t) = 0$.

d. Suppose that $y' = \Lambda y$ is solved with a Runge-Kutta method, using a constant step $h > 0$. Let $r$ be the function from Lemma 4.1. Letting $g = r$, obtain the explicit form of $[r(h\Lambda)]^n, n = 0, 1, \ldots.$

e. Prove that if $h\lambda \in \mathcal{D}$, where $\mathcal{D}$ is the linear stability domain of the Runge-Kutta method, then $\lim_{n \to \infty} y_n = 0$.

**Proof.** a. Denote

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$\Lambda = \lambda I + P.$$

Since we know that $P^k = 0$ for $k = 2, 3, \cdots$, we have

$$\Lambda^n = (\lambda I + P)^n = (\lambda I)^n + n(\lambda I)^{n-1}P = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}. $$

b. As $g$ is an an arbitrary function that is analytic about the origin, $g$ have the Taylor expansion at origin. Then we expand $g(t\Lambda)$ at origin.

$$g(t\Lambda) = g(0) \cdot I + t\Lambda g'(0) + \frac{g''(0)}{2!}(t\Lambda)^2 + \cdots + \frac{g^{(n)}(0)}{n!}(t\Lambda)^n + \cdots$$
By a, we can replace $\Lambda^n$ in the above equation, then

$$g(t\Lambda) = g(0) \cdot I + tg'(0) \left( \begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array} \right) + \frac{g''(0)t^2}{2!} \left( \begin{array}{cc} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{array} \right) + \cdots + \frac{g^{(n)}(0)t^n}{n!} \left( \begin{array}{cc} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{array} \right) + \cdots$$

$$= \left( \begin{array}{cc} g(t\lambda) & tg'(t\lambda) \\ 0 & g(t\lambda) \end{array} \right)$$

\[\text{c. The solution of the differential equation is } y = e^{t\Lambda}y_0, \text{ where } y_0 \text{ is the initial value. As } g(z) = e^z, \text{ then we have}\]

$$e^{t\Lambda} = g(t\Lambda) = \left( \begin{array}{cc} g(t\lambda) & tg'(t\lambda) \\ 0 & g(t\lambda) \end{array} \right) = \left( \begin{array}{cc} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{array} \right)$$

Because $\lambda \in \mathbb{C}^-$, then $\lim_{t \to -\infty} e^{t\lambda} = 0$ and $\lim_{t \to -\infty} te^{t\lambda} = 0$. Then, we have $e^{t\Lambda} \to 0$ when $t \to \infty$. Hence, $\lim_{t \to -\infty} y(t) = 0$.

\[\text{d. By result of a and b, we have}\]

$$[r(h\Lambda)]^n = [g(h\Lambda)]^n = \left( \begin{array}{cc} g(h\lambda) & hg'(h\lambda) \\ 0 & g(h\lambda) \end{array} \right)^n = \left( \begin{array}{cc} (g(h\lambda))^n & n[g(h\lambda)]^{n-1}hg'(h\lambda) \\ 0 & (g(h\lambda))^n \end{array} \right)$$

$$= \left( \begin{array}{cc} [r(h\lambda)]^n & n[r(h\lambda)]^{n-1}hr'(h\lambda) \\ 0 & [r(h\lambda)]^n \end{array} \right).$$

\[\text{e. If } h\lambda \in \mathcal{D}, \text{ where } \mathcal{D} \text{ is the linear stability domain of the Runge-Kutta method, then } |h\lambda| < 1, \text{ so}\]

$$\lim_{n \to \infty} [r(h\lambda)]^n = 0, \quad \lim_{n \to \infty} n[r(h\lambda)]^{n-1} = 0,$$

which means that $[r(h\Lambda)]^n \to 0$ when $n \to \infty$. Hence,

$$\lim_{n \to \infty} y_n = [r(h\Lambda)]^n y_0 = 0.$$ \hfill \Box

2. Exercise 4.6 (a) of the textbook by Iserles.
Evaluate explicitly the function $r$ for the following Runge-Kutta methods.

\[
\begin{array}{c|cc}
0 & 0 & 0 \\
\hline
\frac{1}{2} & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\end{array}
\]
Is this method A-stable?

Solution:

From the RK table, we obtain that

\[
A = \begin{pmatrix} 0 & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad b^T = \begin{bmatrix} 1 & 3 \\ \frac{1}{4} & 4 \end{bmatrix}.
\]

Then we obtain that

\[
r(z) = 1 + z b^T (I - zA)^{-1} 1 \\
= 1 + \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3-z} & \frac{3}{3-z} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{z}{3-z} & \frac{1}{1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
= \frac{z^2 + 4z + 6}{6 - 2z}.
\]

If take \( z = i \), then \( |r(i)| > 1 \). Hence, by lemma 4.3 this method is not A-stable.

3. (Computer Question): Plot the linear stability domains for the three-stage ERK methods given on p.40 (the classic RK) and p.41 (the Nystrom method) of the textbook by Iserles. Are the two domains identical?

Solution: Yes, they are identical. See code tga01.m.

The RK-table of the classic RK:

\[
\begin{array}{c|ccc}
0 & \frac{1}{2} & 1 & 2 \\
\frac{1}{2} & 1 & \frac{1}{6} & \frac{1}{2} & 1 \\
\frac{1}{6} & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\
\end{array}
\]

Then we obtain

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ -1 & 2 & 0 \end{pmatrix}, \quad b^T = \begin{bmatrix} 1 & 2 & 1 \\ \frac{1}{6} & 3 & 6 \end{bmatrix}.
\]

Hence, for the classic RK

\[
r(z) = \frac{z^3}{6} + \frac{z^2}{2} + z + 1.
\]

The RK-table of the Nystrom scheme:

\[
\begin{array}{c|ccc}
0 & \frac{2}{3} & 0 & 0 \\
\frac{2}{3} & 0 & \frac{2}{3} & 0 \\
\frac{1}{4} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\end{array}
\]

Then we obtain

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix}, \quad b^T = \begin{bmatrix} 1 & 3 & 3 \\ \frac{1}{4} & 8 & 8 \end{bmatrix}.
\]
Hence, for the \textit{Nystrom} scheme

\[ r(z) = \frac{z^3}{6} + \frac{z^2}{2} + z + 1. \]

Therefore, these two method have the same linear stability domain:

\[ D = \{ z \in \mathbb{C} : \left| \frac{z^3}{6} + \frac{z^2}{2} + z + 1 \right| < 1 \}, \]

which is the following figure.

![Figure 1: Stability domain for classic RK and Nystrom method.](image)

4. Exercise 4.8 of the textbook by Iserles.
Determine the order of the two-step method

\[ Y_{n+2} - Y_n = h[f(t_{n+2}, Y_{n+2}) + f(t_{n+1}, Y_{n+4}) + f(t_n, Y_n)], n = 0, 1, \ldots \]

Is it $A$-stable?

Solution:
Frow above, we can know that

\[ \rho(w) = w^2 - 1 = \xi^2 + 2\xi, \quad \sigma(w) = \frac{2}{3}(w^2 + w + 1) = \frac{2}{3}\xi^2 + 2\xi + 2 \]

where $\xi = w - 1$. Then

\[ \frac{\rho(w)}{\ln(w)} - \sigma(w) = \frac{1}{3}\xi^2 + 2\xi + 2 + O(\xi^3) - \left( \frac{2}{3}\xi^2 + 2\xi + 2 \right) \]

\[ = -\frac{1}{3}\xi^2 + O(\xi^3) = O(\xi^2). \]

Hence, this method is of order 2.
From the scheme, we can have 
\[ \eta(z, w) = \sum_{m=0}^{2} (a_m - b_mz)w^m = \left( -1 - \frac{2z}{3} \right) w + \left( 1 - \frac{2z}{3} \right) w^2. \]

Then the roots of the equation are 
\[ w_1(z) = \frac{z + \sqrt{9 - 3z^2}}{3 - 2z}, \quad w_2(z) = \frac{z - \sqrt{9 - 3z^2}}{3 - 2z}. \]

Since \([w_i(i)] = 1 (i = 1, 2)\), from lemma 4.8, we know that this method is A-stable.

5. Computer Question: Plot the linear stability domains for both the two-step Adams-Bashforth method and the two-step Adams-Moulton method (i.e., try to reproduce the top two plots in Fig.4.3 of the textbook).

**Solution:**
See code tga02.m.
Two-step Adams-Bashforth method:
\[ y_{n+2} = y_{n+1} + h \left[ \frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right]. \]

We can obtain that 
\[ \eta(z, w) = w^2 - \left( 1 + \frac{3}{2} z \right) w + \frac{1}{2} z \Rightarrow w(z) = \frac{1 + \frac{3}{2} z \pm \sqrt{1 + z + \frac{9}{4} z^2}}{2}. \]

Two-step Adams-Moulton method:
\[ y_{n+2} = y_{n+1} + h \left[ \frac{5}{12} f(t_{n+2}, y_{n+2}) + \frac{2}{3} f(t_{n+1}, y_{n+1}) - \frac{1}{12} f(t_n, y_n) \right]. \]

We can obtain that 
\[ \eta(z, w) = \left( 1 - \frac{5}{12} z \right) w^2 - \left( 1 + \frac{2}{3} z \right) w + \frac{1}{12} z \Rightarrow w(z) = \frac{1 + \frac{2}{3} z \pm \sqrt{1 + z + \frac{21}{36} z^2}}{2 \left( 1 - \frac{5}{12} z \right)}. \]

Hence, we have:
7. Exercise 6.4 of the textbook by Iserles.
Prove that the embedded RK pair:

\[
\begin{array}{c|ccc}
  & \frac{1}{2} & 0 & 0 \\
 0 & 1 & 1 & 2 \\
 1 & 1 & 2 & 1 \\
\end{array}
\]

combines a second-order and a third-order method.

**Solution:**
Consider the 2 stage ERK method

\[ c^T = (0, 1/2), \quad A = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \\ -1 & 2 \end{pmatrix}, \quad b^T = (0, 1). \]

This method is of order 2. In fact, on the one hand, according to (3.7) in the book which is

\[ b_1 + b_2 = 1, \quad b_2c_2 = 1/2, \quad a_{2,1} = c_2, \]

we know that the order is \( \geq 2 \). On the other hand, by applying this ERK method to the scalar equation \( y' = y \), we obtain that the order of this method is \( \leq 2 \). Thus, the method is of order 2.

Consider the 3 stage ERK methods

\[ c^T = (0, 1/2, 1), \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ -1 & 2 & 0 \end{pmatrix}, \quad b^T = (1/6, 2/3, 1/6). \]

This method is of order 3. In fact, from condition on page 40:

\[ b_1 + b_2 + b_3 = 1, \quad b_2c_2 + b_3c_3 = 1/2, \quad b_2c_2^2 + b_3c_3^2 = 1/3, \quad b_3a_3c_2 = 1/6 \]

and similar discussion as above 2-stage case, we can verify that this method is of order 3.

Hence, the given the embedded RK pair combines a second-order and a third-order method.
8. Write down the algorithm of the shooting method (with Newton's method) for solving the BVP: 
\[ y'' = (1 - y^2)y' - y, \text{ with } y(0) = 2 \text{ and } y(1) = 4. \]

Solution:
Input: initial guess for \( y'(0) : z_0 \).
for \( K \) from 0 to \( K_{\text{max}} \).
Solve IVP:
\[
\begin{align*}
    y'' &= (1 - y^2)y' - y, \\
    x'' &= -(2y'y + 1)x + (1 - y^2)x' \\
    y(0) &= 2, y'(0) = z_k \\
    x(0) &= 0, x'(0) = 1
\end{align*}
\]
for \( 0 \leq t \leq 1 \).
If \( |y(1, z_k) - 4| < \text{tolerance} \),
Output: \( \{ y(t_n, z_k) \} \).
end if.
\[
z_{k+1} = z_k - \frac{y(1, z_k) - 4}{x(1)}
\]
end for
Output message: 'Shooting method failed after \( K_{\text{max}} \) trials.'
Appendixes

% linear stability domains for the three-stage ERK methods given on p. 40
% (the classic RK) and p. 41 (the Nystrom method) of textbook by Iserles.

% the classic RK
A1 = zeros(3, 3) + diag([1/2 2], -1) + diag(-1, -2);
b1 = [1/6 2/3 1/6];

% the Nystrom method
A2 = zeros(3, 3) + diag([2/3 2/3], -1);
b2 = [1/4 3/8 3/8];

[X, Y] = meshgrid(linspace(-3, 1), linspace(-3, 3));
Z = X + Y * 1i;

xi_1 = 1;
xi_2 = 1 + Z * A1(2, 1);
xi_3 = 1 + Z * A1(3, 1) + Z * A1(3, 2) * xi_2;
phi1 = 1 + Z * (b1(1) * xi_1 + b1(2) * xi_2 + b1(3) * xi_3);

xi1 = 1;
xi2 = 1 + Z * A2(2, 1);
xi3 = 1 + Z * A2(3, 1) + Z * A2(3, 2) * xi2;
phi2 = 1 + Z * (b2(1) * xi1 + b2(2) * xi2 + b2(3) * xi3);

figure(1)
contourf(X, Y, 1-abs(phi1), [0 0], 'LineWidth', 1);
set(gca, 'FontSize', 10, 'CLim', [0 1]);
colormap(spring);
hold on;
plot([-3 1], [0 0], '-k', 'LineWidth', 1);
plot([0 0], [-3 3], '-k', 'LineWidth', 1);
title('linear stability domain for classic RK');

figure(2)
contourf(X, Y, 1-abs(phi2), [0 0], 'LineWidth', 1);
set(gca, 'FontSize', 10, 'CLim', [0 1]);
colormap(autumn);
hold on;
plot([-3 1], [0 0], '-k', 'LineWidth', 1);
plot([0 0], [-3 3], '-k', 'LineWidth', 1);
title('linear stability domain for Nystrom method');

% linear stability domains for the 2 step AB and 2 step AM:
% the 2-step AB
[X,Y] = meshgrid(linspace(-1.5,1), linspace(-1,1));
Z = X+Y*1i;
a=ones(size(Z));
b=ones(size(Z))-3/2*Z;
c=Z/2;
omega1=(-b+sqrt(b.^2-4*a.*c))/(2*a);
omega2=(-b-sqrt(b.^2-4*a.*c))/(2*a);

% the 2-step AM
[XX,YY] = meshgrid(linspace(-7,1), linspace(-4,4));
ZZ = XX+YY*1i;
aa=ones(size(ZZ))-5/12*ZZ;
bb=ones(size(ZZ))-2/3*ZZ;
c=ZZ/12;
omega3=(-bb+sqrt(bb.^2-4*aa.*cc))/(2*aa);
omega4=(-bb-sqrt(bb.^2-4*aa.*cc))/(2*aa);

figure(1)
contourf(X,Y,1-max(abs(omega1),abs(omega2)), [0 0], 'LineWidth', 1);
set(gca, 'FontSize', 10, 'CLim', [0 1]);
colormap(spring);
hold on;
plot([-1.5 1], [0 0], '--k', 'LineWidth', 1);
plot([0 0], [-1 1], '--k', 'LineWidth', 1);
title('linear stability domain for 2-step AB')

figure(2)
contourf(XX,YY,1-max(abs(omega3),abs(omega4)), [0 0], 'LineWidth', 1);
set(gca, 'FontSize', 10, 'CLim', [0 1]);
colormap(autumn);
hold on;
plot([-7 1], [0 0], '--k', 'LineWidth', 1);
plot([0 0], [-4 4], '--k', 'LineWidth', 1);
title('linear stability domain for 2-step AM')