Homework Assignments – Math 122-01/02 – Spring 2015

Assignment 1
Due date: Friday, January 23
Section 5.1, Page 159: #1-4, 10, 11, 14;
Section 5.2, Page 163: Find the slope and y-intercept, and then plot the line in problems: #2, 4;
Find vertex, axis of symmetry, focus, focal width, and then plot the parabola in:
#6, 8;
Section 4.1, Page 143: #2, 3, 7, 8;
Section 4.2, Page 151: #3, 4, 6, 13, 14, 15, 29, 30, 32, 34;

Required Additional Exercises #1 – 5 (below).

Instructions
In problems 1-3, plot the function and indicate its domain and range.

1. \( f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x < 2 \\ 1 + x & \text{if } 2 \leq x \leq 5 \end{cases} \)
2. \( f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x < 2 \\ 3 - x^2 & \text{if } 2 \leq x \leq 5 \end{cases} \)
3. \( f(x) = \frac{|x|}{x} \) for all \( x \neq 0 \).

4. Determine the value of \( f[f(x)] \) for the function \( f \) in problem 3 above.

5. Find the exact coordinates of the square DEFG inscribed in \( \triangle ABC \) given that \( A = (0, 0) \), \( B = (2, 4) \) and \( C = (5, 0) \).

Assignment 2
Due date: Monday, February 2
Section 23.1, Page 745: #2, 3, 7, 8, 9,14, 15, 17, 18, 23, 25, 26, 38.

Required Additional Exercises #1 – 4 (below).

1. (a) Sketch a graph of the function: \( f(x) = \begin{cases} 1 - x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \)
(b) Use the formula for \( f \) from part (a) to determine the value of each of the following limits, if they exists.
   (i) \( \lim_{x \to 0^+} f(x) \) (ii) \( \lim_{x \to 0^-} f(x) \) (iii) \( \lim_{x \to 0^+} f(x) \)
   \( \lim_{x \to 0^-} f(x) \) \( \lim_{x \to 0^+} f(x) \) \( \lim_{x \to 0^-} f(x) \)

2. (a) Sketch a graph of the function: \( f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ 4 & \text{if } x = 1 \\ 4 - x & \text{if } x > 1 \end{cases} \)
(b) Use the formula for \( f \) from part (a) to determine the value of each of the following limits, if they exists.
   (i) \( \lim_{x \to -1^+} f(x) \) (ii) \( \lim_{x \to -1^-} f(x) \) (iii) \( \lim_{x \to 1^-} f(x) \) (iv) \( \lim_{x \to 1^+} f(x) \) (v) \( \lim_{x \to -1} f(x) \) (vi) \( \lim_{x \to 1} f(x) \)

3. A graph of the function \( y = g(x) \) is shown in the figure.
   (a) State the domain and range of \( g \).
   (b) Use the graph to determine each of the limits:
      (i) \( \lim_{x \to -1} g(x) \) (ii) \( \lim_{x \to -1} g(x) \) (iii) \( \lim_{x \to -1} g(x) \)
      (iv) \( \lim_{x \to 0} g(x) \) (v) \( \lim_{x \to 0} g(x) \) (vi) \( \lim_{x \to 0} g(x) \).
4. (a) Write a piecewise definition for the function \( y = g(x) \) shown in the figure.
(b) State the domain and range of \( g \).
(c) Determine the limits: \( \lim_{{x \to 1^-}} g(x) \), \( \lim_{{x \to 1^+}} g(x) \).

Assignment 3
Due date: Monday, February 9
Section 23.3, Page 758:
Use the method of Example 14, Page 751, to find the derivative of \( y = f(x) \) in each of the following problems: #1, 6, 8, 9, 11, 12.

To find the slope of the graph of \( y = f(x) \) at the given value, \( x = a \), evaluate the limit:
\[
\frac{dy}{dx} \bigg|_a = f'(a) = \lim_{{\Delta x \to 0}} \frac{f(a + \Delta x) - f(a)}{\Delta x}
\]
in each of the following problems: #13, 15, 17, 19.

Required Additional Exercises  #1 – 7 (below).

Instructions
In problems 1-4, estimate the slope of the curve at the point \( P \).

1.  

2.  

3.  

4.  

5. Assign one of the following descriptors to each point \( A, B, C, D, E, F \) in the figure:
- large positive slope
- small positive slope
- zero slope
- small negative slope
- large negative slope.

6. Estimate the value of \( f'(1) \) in the figure below.

7. Find the equation of the tangent line to \( f(x) \) at the point \( P \).
1. A stone is thrown vertically upward into the air with an initial velocity of 96 ft/s. On Mars, the height of the stone above the ground after $t$ seconds is $h = 96t - 6t^2$, and on Earth, $h = 96t - 16t^2$. How much higher will the stone travel on Mars than on Earth?

2. The graph displays the position $s = f(t)$ of a freight train locomotive $t$ hours after 5:00pm relative to its starting point $s = 0$, where $s$ is measured in miles.

(a) Describe the speed of the train. Specifically, when is it speeding up and when is it slowing down?

(b) At approximately what time of day is it traveling the fastest? The slowest? Why?

In problems 3 - 8, determine whether the given functions is:

(a) continuous at $x = 1$ \iff $\lim_{x \to 1} f(x) = f(1)$.

(b) differentiable at $x = 1$ \iff $f$ is continuous at $x = 1$ and $\lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} 
= f'(1) \equiv \lim_{\Delta x \to 0} \frac{\Delta f(1)}{\Delta x}$

(c) Then sketch a graph of $y = f(x)$.

3. $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

4. $f(x) = \begin{cases} 2x-1 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$

5. $f(x) = \begin{cases} x+2 & \text{if } -1 \leq x \leq 1 \\ 3x & \text{if } 1 < x \leq 3 \end{cases}$

6. $f(x) = \begin{cases} 1-x^2 & \text{if } -1 \leq x < 1 \\ x-1 & \text{if } 1 \leq x \leq 3 \end{cases}$

7. $f(x) = \begin{cases} x^3 & \text{if } -1 \leq x < 1 \\ 3x-2 & \text{if } 1 \leq x \leq 3 \end{cases}$

8. $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

In problems 9 and 10, $f$ is defined for all $x$ except for one value. If possible, define $f(x)$ at that exceptional point in order to make it continuous there.

9. $f(x) = \frac{x^2 + x - 12}{x + 4}, x \neq -4$

10. $f(x) = \frac{(6+x)^2 - 36}{x}, x \neq 0$

11. For what number $a$ will the function $f(x) = \begin{cases} (x-a)^2 + a & \text{if } x \leq 1 \\ a & \text{if } x > 1 \end{cases}$ be both continuous and differentiable at $x = 1$? Sketch the graph of $y = f(x)$ using this particular value of $a$.

12. Explain in words why the function described in each case below is either continuous everywhere on its domain, or if instead, it possesses discontinuities. Then sketch a rough graph of the function.

(a) The temperature, $T$, (as a function of time) of a pitcher of ice water left to stand on a table for several hours in a 70$^\circ$F room.

(b) The cost, $C$, (as a function of time) for parking at the Millennium Park Garage, 5 S. Columbus Drive, in downtown Chicago for anywhere from 0 to 24 hours. The parking rates are:

- $23$ for 0 to 3 hours
- $26$ for 3 to 8 hours
- $28$ for 8 to 12 hours
- $30$ for 12 to 24 hours.
Assignment 5

Due date: Friday, February 27
Section 23.4, Page 764: #1, 2, 8, 13, 18, 20, 22, 28, 35, 46, 49, 61.
Section 23.5, Page 769: #2, 9, 13, 17, 20, 31.
Section 23.6, Page 775: #2, 13, 14, 17, 21, 25, 30, 31, 33, 37, 38, 43.

Required Additional Exercises #1 – 2 (below).

1. The acceleration due to gravity, \( g \), varies with height above the surface of the earth in a certain way. If you go down below the surface of the earth, then \( g \) varies in a different way. It can be shown that \( g \) is given by:

\[
\begin{align*}
g(r) &= \begin{cases} 
\frac{GM}{r^2} & \text{for } 0 \leq r < R \vspace{0.1cm} \\
\frac{GM}{R^2} & \text{for } r \geq R
\end{cases}
\end{align*}
\]

where \( R \) is the radius of the earth, \( M \) is the mass of the earth, \( G \) is the gravitational constant, and \( r \) is the distance to the center of the earth.

(a) Sketch a graph of \( g \) against \( r \) for \( r > 0 \).

(Note: Since \( r \) is the independent variable, you should measure \( r \) along the horizontal axis.)

(b) Is \( g \) continuous at \( r = R \)? Explain your answer.

(c) Compute the piecewise formula for \( g'(r) \) and investigate whether or not \( g \) is differentiable at \( r = R \).

2. Look at the graph of \( f(x) = (x^2 + 0.0001)^{1/2} \) in the figure. This graph appears to have a sharp corner at \( x = 0 \). Does it? Is \( f \) differentiable at \( x = 0 \)? Justify your answers.

Assignment 6

Due date: Wednesday, March 11
Section 23.7, Page 782: #15, 16, 20, 21, 23, 24, 25, 26, 27, 28, 39.

Required Additional Exercises #1 – 16 (below).

Instructions: In problems 1 – 15, find the derivative of \( y \) in each case.

1. \( y = (2x - 7)^3 \)
2. \( y = (3x^2 + 1)^3 \)
3. \( y = 3x(4 - 9x)^3 \)
4. \( y = (3 + x)^2 \left( 1 - x^2 \right)^3 \)
5. \( y = (9 - x^2)^{3/2} \)
6. \( y = 3x - 2x^2 \)
7. \( y = \sqrt[3]{9x^2 + 2x + 7} \)
8. \( y = \sqrt[3]{x^3 - 2x + 1} \)
9. \( y = \frac{2x^3 - 1}{x^2} \)
10. \( y = \frac{x + 1}{x - 1} \)
11. \( y = \frac{6x - 5}{x^2 + 1} \)
12. \( y = \frac{2x + 3}{3x^2 - 2x} \)
13. \( y = \left( x + \frac{1}{2} \right)^4 \)
14. \( y = x^2 \left( x - 1 \right)^4 \)
15. \( y = \frac{(4x - 1)^3}{2x^2 + 1} \)
16. Use calculus to find the coordinates of the two points along the graph of \( y = x^2 \) whose tangent lines pass through point \( P = (1, -3) \). Then sketch a graph which displays both the parabola and its tangent lines which intersect at \( P \).
Assignment 7
Due date: Wednesday, March 25

Exercises # 1 – 4 (below).

Refer to the lecture notes: 
http://mypages.iit.edu/~maslanka/ImplicitDiff.pdf
and
http://mypages.iit.edu/~maslanka/PolarConversion.pdf

for details on implicit differentiation, the conversion of equations from rectangular to polar coordinates, and plotting in polar coordinates.

1. (a) Use implicit differentiation to find the slope and inclination of the tangent line to the graph of the cardioid: 
\((x^2+y^2)^2=x^2+y^2\)

at the points \(P = (1, 0)\) and \(Q = (-1, 0)\).

(b) Set \(x = r \cos \theta\) and \(y = r \sin \theta\) in the equation from (a) and simplify it, to obtain the polar equation of the cardioid: 
\(r = 1 - \sin \theta\).

(c) Sketch a careful graph which displays the cardioid and its tangent lines at \(P\) and \(Q\).

2. (a) Use implicit differentiation to find the slope and inclination of the tangent line to the graph of the circle: 
\((x-1)^2 + y^2 = 1\)

at the point \(P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\).

(b) Set \(x = r \cos \theta\) and \(y = r \sin \theta\) in the equation from (a) and simplify it, to obtain the polar equation of the circle: 
\(r = 2 \cos \theta\).

(c) Sketch a careful graph which displays the circle and its tangent line at \(P\).

3. (a) Use implicit differentiation to find the slope and inclination of the tangent line to the graph of the parabola: 
\(y^2 = 4x\)

at the points \(P = (1, 2)\) and \(Q = (4, -4)\).

(b) Set \(x = r \cos \theta\) and \(y = r \sin \theta\) in the equation from (a) and simplify it, to obtain the polar equation of the cardioid: 
\(r = 4 \cot \theta \csc \theta\).

(c) Sketch a careful graph which displays the parabola and its tangent lines at \(P\) and \(Q\).

4. (a) Use implicit differentiation to find \(\frac{dy}{dx}\) for the lemniscate: 
\((x^2+y^2)^2=x^2-y^2\).

(b) Set \(x = r \cos \theta\) and \(y = r \sin \theta\) in the equation from (a) and simplify it, to obtain the polar equation of the lemniscate.

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Extra Credit Exercise

In each of the following problems, \(f\) is differentiable on an interval containing the arbitrary points \(a\) and \(b\), \(P = (a, f(a))\) and \(Q = (b, f(b))\) are distinct points on the graph of \(f\), and \(c\) is the \(x\)-coordinate of the point where the tangents tangent to the curve at \(P\) and \(Q\) intersect, assuming that the tangent lines are not parallel (as in the figure shown to the right).

(a) If \(f(x) = x^2\), draw a figure which displays the graph of \(f\), two points \(P = (a, f(a))\) and \(Q = (b, f(b))\) on its graph, and the point \(R = (c, f(c))\) where the tangents to the graph of \(f\) at \(P\) and \(Q\) intersect. Then prove that \(c = a + b\), this is the arithmetic mean of \(a\) and \(b\).

(b) If \(f(x) = \sqrt[3]{x}\), draw a figure which displays the graph of \(f\), two points \(P = (a, f(a))\) and \(Q = (b, f(b))\) on its graph, (where \(a, b > 0\)), and the point \(R = (c, f(c))\) where the tangents to the graph of \(f\) at \(P\) and \(Q\) intersect. Then prove that \(c = \sqrt[3]{ab}\), this is the geometric mean of \(a\) and \(b\).

(c) If \(f(x) = \frac{1}{x}\), draw a figure which displays the graph of \(f\), two points \(P = (a, f(a))\) and \(Q = (b, f(b))\) on its graph, (where \(a, b > 0\)), and the point \(R = (c, f(c))\) where the tangents to the graph of \(f\) at \(P\) and \(Q\) intersect. Then prove that \(c = \frac{2ab}{a+b}\), this is the harmonic mean of \(a\) and \(b\).
Assignment 8

Due date : Wednesday, April 1
Section 25.4 Page 835: #6, 8, 10, 13.
Required Additional Exercises # 1 – 16 (below).

Refer to the lecture notes (http://mypages.iit.edu/~maslanka/ExtremeValues.pdf) for details on relative and absolute extreme values and associated theorems and results.

On Relative and Absolute Extreme Values and Critical Numbers
1. Explain the difference between an absolute minimum and a relative minimum value.
2. Suppose that \( f \) is a continuous function defined on the closed interval \([a, b]\).
   (a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for \( f \)?
   (b) What steps would you take to find those absolute extreme values?
In problem 3 – 5, sketch the graph of a function \( f \) that is continuous on \([a, b]\) and has the given properties.
3. Absolute minimum at 2, absolute maximum at 3, relative minimum at 4.
4. Absolute maximum at 5, absolute minimum at 2, relative maximum at 3, relative minima at 2 and 4.
5. \( f \) has no relative maximum or minimum but 2 and 4 are critical numbers.

Recall that \( c \) is a critical number for \( f \) \( \iff \) \( c \) is in the interior of \( \text{dom}(f) \) and either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

6. (a) Sketch the graph of a function that has a relative maximum at 2 and is differentiable at 2.
   (b) Sketch the graph of a function that has a relative maximum at 2 and is continuous but not differentiable at 2.
   (c) Sketch the graph of a function that has a relative maximum at 2 and is discontinuous at 2.

In problems 7 – 10 sketch the graph of \( f \) over the indicated interval. Then locate all absolute and relative maximum and minimum values of \( f \) that exist.
7. \( f(x) = 8 - 3x, \ x > 1 \)
8. \( f(x) = x^5, \ -1 \leq x \leq 2 \)
9. \( f(x) = x^3, \ -1 \leq x < 2 \)
10. \( f(x) = 1/x, \ 0 < x \leq 2 \)

In problems 11 – 16 find all critical numbers for the function, all the intervals on which \( f \) is increasing or decreasing, and all relative and absolute extreme values. Then use this information to help you sketch a graph of \( y = f(x) \).
11. \( f(x) = x^4 - 2x^2 + 1 \)
12. \( f(x) = x^5 - 2x^3 \)
13. \( f(x) = x^3 + 3x^2 - 24x \)
14. \( f(x) = |2x + 3| \)
15. \( f(x) = x^3 - 3x + 1 \)
16. \( f(x) = \sqrt[3]{x} \cdot (8 - x) \)

Assignment 9

Due date : Wednesday, April 8
Section 25.4 Page 836: #17, 21, 23, 29.
Section 29.1 Page 947: #2, 3, 4, 7, 11, 12, 19, 26, 29, 36.
Required Additional Exercises # 1 – 7 below.

In problems 1 – 6 find all extreme points and all inflection points for the graph of the function. Then find all the intervals on which \( f \) is increasing, decreasing, concave up, and concave down and sketch a graph of \( y = f(x) \).
(Note that problems 1 – 5 were initially examined in the previous assignment.)
1. \( f(x) = x^3 - 2x^2 + 1 \)
2. \( f(x) = x^4 - 2x^3 \)
3. \( f(x) = x^3 + 3x^2 - 24x \)
4. \( f(x) = x^3 - 3x + 1 \)
5. \( f(x) = \sqrt[3]{x} (8 - x) \)
6. \( f(x) = 2x^{1/3} + x^{2/3} \)

7. A marble of radius \( r, \ 0 < r < 4, \) is dropped into a container having the shape of a circular cylinder. The base of the container has radius 4. What is the radius of the marble that requires the most water to cover it completely?

Recall that \( V_{sphere} = \frac{4}{3} \pi r^3 \).
Assignment 10

**Due date:** Monday, April 20

Section 29.2 Page 951: #1, 4, 9, 12, 13, 21, 26, 27.

**Required Additional Exercises**  # 1 – 3 below.

In problems 1 – 3 do the following:

(a) Sketch a rectangular plot of the given equation in the \( r \) \(-\) plane over the indicated \( \theta \) interval.

(b) Sketch a polar graph of the equation in the \( xy \) \(-\) plane, where \( x = r \cos \theta \), \( y = r \sin \theta \).

(c) Find the slope and inclination of the tangent line to the polar curve at the point

\[
P = (x, y) = (r \cdot \cos(\frac{\theta}{4}), r \cdot \sin(\frac{\theta}{4}))
\]

by using the formula:

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.
\]

(Refer to the notes at [http://mypages.iit.edu/~maslanka/PolarTangents.pdf](http://mypages.iit.edu/~maslanka/PolarTangents.pdf) for reference on this topic.)

1. \( r = \frac{\pi}{6} ; \frac{\pi}{6} \leq \theta \leq 2\pi \)
2. \( r = 1 + \sin \theta ; 0 \leq \theta \leq 2\pi \)
3. \( r = 2 \sin (3\theta) ; 0 \leq \theta \leq \pi \)

Assignment 11

**Due date:** Monday, April 27

Section 26.1 Page 850: #3, 7, 12, 19, 22, 27, 31, 35, 36, 38, 39.

Section 26.2 Page 857: #1, 3, 6, 7, 8, 10, 17, 20, 21, 24.

Section 26.3 Page 862: #2, 3, 8, 13, 14, 17.

Section 29.7 Page 973: #1, 2, 9, 10, 19.

**Required Additional Exercises**  # 1 – 3 below.

In problems 1 – 3 the graph of a function \( f \) is displayed. Sketch the graph of an antiderivative function, \( F \), in each case. Include an analysis of the intervals on which \( F \) is increasing/decreasing and concave up/down based on your knowledge of \( f = F' \).

1. \[ y \]
2. \[ y \]
3. \[ y \]

Assignment 12

**Due date:** Wednesday, May 6

Section 26.4 Page 866: #2, 5, 6, 9, 11, 12, 13.

Section 26.6 Page 874: #1, 6, 13.

Section 29.7 Page 974: #27, 29.

**Required Additional Exercises**  # 1 – 7 below.
1. A uniform horizontal beam of length \( L \) having a simple point support at its left end and a fixed support at its right end will be distorted, due to its own weight, into a curve:
\[ y = f(x) \]
as shown in the figure below.

This curve is called the **deflection curve** of the beam and it satisfies the differential equation:
\[
E \cdot I \frac{d^4y}{dx^4} = w
\]
along with the four boundary conditions:
\[
y(0) = 0, \quad y''(0) = 0 \\
y(L) = 0, \quad y'(L) = 0
\]
By substituting each boundary condition into either equation (2), (3) or (4) from page 2 of the article:
http://mypages.iit.edu/~maslanka/BeamDeflection.pdf
and then solving for the coefficients, \( C_i, \ i = 1, 2, 3, 4 \), verify that the beam’s equation is:
\[
y = \frac{w}{48EI} \left[ 2x^4 - 3Lx^3 + L^3 x \right]
\]

**Mean Value Theorem:**
If \( F \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\) then there must exist at least one number \( c \) in \((a, b)\) for which
\[
F'(c) = \frac{F(b) - F(a)}{b - a}
\]

In problems 2 and 3 do the following:

(a) Apply the Mean Value Theorem to the function:
\[ F(x) = x^3 - 3x \]
on the indicated interval \( I \) to find all numbers, \( c \), in the interior of \( I \), which satisfy the conclusion of this theorem. I.e., find all numbers \( c \) interior to \( I \) for which the average change in \( F \) over \( I \) is equal to the instantaneous rate of change in \( F \) at \( c \).

(b) Sketch a graph of the function \( y = F(x) \) on the interval \( I \) and interpret your results from part (a) geometrically in terms of the appropriate tangent and secant lines to your graphs.

2. \( I = [0, +2] \)  
3. \( I = [-1, +1] \)

In problems 4 – 7 use the definite integral to find the area of the shaded region in each case.
Exam 3 Review Topics: (http://mypages.iit.edu/~maslanka/Math122Rvw3Tpcs.pdf)
Exam 3 Review Problem Set: (http://mypages.iit.edu/~maslanka/Math122Rvw3.pdf)

(Final) Exam 3 dates and times:
- Math 122-01, 8:00am – 10:00 am, Wednesday, May 6, Room 106 E1
- Math 122-02, 10:30am – 12:30 pm, Wednesday, May 6, Room 106 E1.