Problems On the Legendre Equation and Legendre Polynomials

Problems 1–6 deal with the Legendre equation:

\[(1 - x^2) y'' - 2x y' + \alpha (\alpha + 1) y = 0.\]  \hspace{1cm} (1)

In this DE, \(\alpha\) denotes a real constant.

Observe that it is really only necessary to consider its solution in the case where \(\alpha > -1\), since when \(\alpha \leq -1\) then the substitution \(\alpha = - (1 + \beta)\) where \(\beta \geq 0\) leads to the Legendre equation:

\[(1 - x^2) y'' - 2x y' + \beta (\beta + 1) y = 0\]  \hspace{1cm} (2)

where \(\alpha (\alpha + 1) = \beta (\beta + 1)\). Thus, equations (1) and (2) must have identical solutions.

Exercises

1. Apply the Existence Theorem for Power Series Solutions About Ordinary Points to establish that the Legendre Equation (1) has two linearly independent solutions of the form:

\[y(x) = \sum_{n=0}^{\infty} c_n x^n\] for any choice of \(\alpha\). Then specify the guaranteed radius of convergence of each such solution due to this theorem.

2. Show that the two linearly independent solutions of the Legendre equation (1) centered at zero are:

\[y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \times \frac{[(\alpha+1)(\alpha+2)\ldots(\alpha+2n-1)] \times [\alpha(\alpha-2)(\alpha-4)\ldots(\alpha-2n+2)]}{(2n)!} x^{2n}\]

and

\[y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \times \frac{[(\alpha+2)(\alpha+4)\ldots(\alpha+2n)] \times [(\alpha-1)(\alpha-3)\ldots(\alpha-2n+1)]}{(2n+1)!} x^{(2n+1)}\].

Show the details of your work and include your derivation of the recurrence formula for the coefficients in the series.

3. (a) Show that if \(\alpha\) is either zero or a positive even integer, \(2n\), then the series solution \(y_1\) reduces to a polynomial of degree \(2n\) containing only even powers of \(x\). Find the polynomials corresponding to \(\alpha = 0, 2, \text{ and } 4\).

(b) Show that if \(\alpha\) is a positive odd integer, \(2n + 1\), then the series solution \(y_2\) reduces to a polynomial of degree \(2n + 1\) containing only odd powers of \(x\). Find the polynomials corresponding to \(\alpha = 1, 3, \text{ and } 5\).
4. The **Legendre polynomial** $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

(a) Using the results of problem 3, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.

(b) Plot the graphs of $P_0(x), \ldots, P_5(x)$ in the same coordinate plane over the interval $-1 \leq x \leq 1$.
You may either produce the graphs by hand drawing or by using Maple, Matlab or Mathematica. On your graph, indicate the exact location of the zeros (i.e. the $x$-intercepts) of each of these five polynomials and use the **First Derivative Test** to locate their extreme points.

5. Show that the Legendre equation (1) can also be written as $[(1 - x^2)y']' = -\alpha (\alpha + 1)y$.
Then it follows that $[(1 - x^2)P_n'(x)]' = -n (n + 1) P_n(x)$
and $[(1 - x^2)P_m'(x)]' = -m (m + 1) P_m(x)$.
By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, and then integrating by parts, show that
\[ \int_{-1}^{1} P_n(x) \cdot P_m(x) \, dx = 0 \quad \text{whenever} \quad n \text{ and } m \text{ denote integers and } n \neq m. \]
This property of the Legendre polynomials is known as the **orthogonality property**.

6. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace’s equation (the potential equation) in spherical coordinates, we encounter the equation:
\[ \frac{d^2 y}{d\varphi^2} + \cot \varphi \frac{dy}{d\varphi} + n (n + 1) y = 0, \quad 0 < \varphi < \pi, \]
where $n$ is a positive integer. Show that the change of variable $x = \cos \varphi$ leads to the Legendre equation with $\alpha = n$.

**Hint:** By the Chain Rule: $\frac{dy}{d\varphi} = \frac{dy}{dx} \cdot \frac{dx}{d\varphi} = -\sin \varphi \cdot \frac{dy}{dx}$.

Now express $\frac{d^2 y}{d\varphi^2}$ in terms of $\varphi, \frac{dy}{dx},$ and $\frac{d^2 y}{dx^2}$.

Then substitute these results back into the above DE to obtain Legendre’s equation.