3. EVALUATION OF TRIGONOMETRIC FUNCTIONS

In this section, we obtain values of the trigonometric functions for quadrantal angles, we introduce the idea of reference angles, and we discuss the use of a calculator to evaluate trigonometric functions of general angles.

In Definition 2.1, the domain of each trigonometric function consists of all angles $\theta$ for which the denominator in the corresponding ratio is not zero. Because $r > 0$, it follows that $\sin \theta = y/r$ and $\cos \theta = x/r$ are defined for all angles $\theta$. However, $\tan \theta = y/x$ and $\sec \theta = r/x$ are not defined when the terminal side of $\theta$ lies along the $y$ axis (so that $x = 0$). Likewise, $\cot \theta = x/y$ and $\csc \theta = r/y$ are not defined when the terminal side of $\theta$ lies along the $x$ axis (so that $y = 0$). Therefore, when you deal with a trigonometric function of a quadrantal angle, you must check to be sure that the function is actually defined for that angle.

**Example 3.1**

Find the values (if they are defined) of the six trigonometric functions for the quadrantal angle $\theta = 90^\circ$ (or $\theta = \pi/2$).

In order to use Definition 1, we begin by choosing any point $(0, y)$ with $y > 0$, on the terminal side of the $90^\circ$ angle (Figure 1). Because $x = 0$, it follows that $\tan 90^\circ$ and $\sec 90^\circ$ are undefined. Since $y > 0$, we have

$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + y^2} = |y| = y.$$ 

Therefore,

$$\sin 90^\circ = \frac{y}{r} = \frac{y}{y} = 1 \quad \cos 90^\circ = \frac{x}{r} = \frac{0}{y} = 0 \quad \csc 90^\circ = \frac{r}{y} = \frac{y}{y} = 1 \quad \cot 90^\circ = \frac{y}{x} = \frac{y}{0} = 0.$$ 

The values of the trigonometric functions for other quadrantal angles are found in a similar manner. The results appear in Table 3.1. Dashes in the table indicate that the function is undefined for that angle.

**Table 3.1**

<table>
<thead>
<tr>
<th>$\theta$ degrees</th>
<th>$\theta$ radians</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
<th>$\cot \theta$</th>
<th>$\sec \theta$</th>
<th>$\csc \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>—</td>
<td>1</td>
<td>—</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
<td>—</td>
<td>0</td>
<td>—</td>
<td>1</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>$\pi$</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>—</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$270^\circ$</td>
<td>$\frac{3\pi}{2}$</td>
<td>-1</td>
<td>0</td>
<td>—</td>
<td>0</td>
<td>—</td>
<td>-1</td>
</tr>
<tr>
<td>$360^\circ$</td>
<td>$2\pi$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>—</td>
<td>1</td>
<td>—</td>
</tr>
</tbody>
</table>
It follows from Definition 2.1 that the values of each of the six trigonometric functions remain unchanged if the angle is replaced by a coterminal angle. If an angle exceeds one revolution or is negative, you can change it to a nonnegative coterminal angle that is less than one revolution by adding or subtracting an integer multiple of $360^\circ$ (or $2\pi$ radians).

For instance,

\[
\sin 450^\circ = \sin(450^\circ - 360^\circ) = \sin 90^\circ = 1.
\]

\[
\sec 7\pi = \sec \left( 7\pi - \left( 3 \times 2\pi \right) \right) = \sec \pi = -1.
\]

\[
\cos (-660^\circ) = \cos (-660^\circ + (2 \times 360^\circ)) = \cos 60^\circ = \frac{1}{2}.
\]

In Examples 3.2 and 3.3, replace each angle by a nonnegative coterminal angle that is less than one revolution and then find the values of the six trigonometric functions (if they are defined).

**Example 3.2**

$\theta = 1110^\circ$

By dividing 1110 by 360, we find that the largest integer multiple of $360^\circ$ that is less than 1110 is $3 \times 360^\circ = 1080^\circ$. Thus,

\[1110^\circ - (3 \times 360^\circ) = 1110^\circ - 1080^\circ = 30^\circ.\]

(Or we could have started with $1110^\circ$ and repeatedly subtracted 360° until we obtained 30°.) It follows that

\[
\sin 1110^\circ = \sin 30^\circ = \frac{1}{2},
\]

\[
\csc 1110^\circ = \csc 30^\circ = 2,
\]

\[
\cos 1110^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2},
\]

\[
\sec 1110^\circ = \sec 30^\circ = \frac{2\sqrt{3}}{3},
\]

\[
\tan 1110^\circ = \tan 30^\circ = \frac{\sqrt{3}}{3},
\]

\[
\cot 1110^\circ = \cot 30^\circ = \sqrt{3}.
\]

**Example 3.3**

$\theta = -\frac{5\pi}{2}$

We repeatedly add $2\pi$ to $-\frac{5\pi}{2}$ until we obtain a nonnegative coterminal angle:

\[-\frac{5\pi}{2} + 2\pi = -\frac{\pi}{2} \quad \text{(still negative)}\]

\[-\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}.\]

Therefore, by Table 3.1 for quadrant angles,

\[
\sin \left( -\frac{5\pi}{2} \right) = \sin \left( \frac{3\pi}{2} \right) = -1, \quad \cot \left( -\frac{5\pi}{2} \right) = \cot \left( \frac{3\pi}{2} \right) = 0.
\]

\[
\cos \left( -\frac{5\pi}{2} \right) = \cos \left( \frac{3\pi}{2} \right) = 0, \quad \csc \left( -\frac{5\pi}{2} \right) = \csc \left( \frac{3\pi}{2} \right) = -1.
\]

and both $\tan \left( -\frac{5\pi}{2} \right)$ and $\sec \left( -\frac{5\pi}{2} \right)$ are undefined.
<table>
<thead>
<tr>
<th>θ degrees</th>
<th>θ radians</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
<th>cot θ</th>
<th>sec θ</th>
<th>csc θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \sqrt{3} )</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>1</td>
<td>1</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>60°</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \sqrt{3} )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>2</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
</tr>
</tbody>
</table>

**Table 3.2**

**Figure 3.2**

(a) \( \theta = \theta_R \)  
(b) \( \theta_R = 180° - \theta \)  
\( \theta_R^{\text{rad}} = \pi - \theta^{\text{rad}} \)

(c) \( \theta_R = 360° - \theta \)  
\( \theta_R^{\text{rad}} = 2\pi - \theta^{\text{rad}} \)
Example 3.4

Find the reference angle \( \theta_R \) for each angle \( \theta \).

(a) \( \theta = 60^\circ \)  \( \theta_R = \theta = 60^\circ \).
(b) \( \theta = \frac{3\pi}{4} \)  \( \theta_R = \pi - \theta = \pi - \frac{3\pi}{4} = \frac{\pi}{4} \).
(c) \( \theta = 210^\circ \)  \( \theta_R = \theta - 180^\circ = 210^\circ - 180^\circ = 30^\circ \).
(d) \( \theta = \frac{5\pi}{3} \)  \( \theta_R = 2\pi - \theta = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3} \).

The value of any trigonometric function of any angle \( \theta \) is the same as the value of the function for the reference angle, \( \theta_R \), except possibly for a change of algebraic sign.

Thus,
\[
\sin \theta = \pm \sin \theta_R, \quad \cos \theta = \pm \cos \theta_R,
\]
and so forth. You can always determine the correct algebraic sign by considering the quadrant in which \( \theta \) lies.

Section 3 Problems

In problems 1 and 2, find the values (if they are define) of the six trigonometric functions of the given quadrantal angles. (Do not use a calculator.)

1. (a) \( 0^\circ \)  \( \theta = 0^\circ \)  \( \theta_R = 0^\circ \).
   (b) \( 180^\circ \)  \( \theta = 180^\circ \)  \( \theta_R = 180^\circ \).
   (c) \( 270^\circ \)  \( \theta = 270^\circ \)  \( \theta_R = 270^\circ \).
   (d) \( 360^\circ \)  \( \theta = 360^\circ \)  \( \theta_R = 360^\circ \).

   [When you have finished, compare your answers with the results in Table 3.1]

2. (a) \( 5\pi \)  \( \theta = 5\pi \)  \( \theta_R = \pi \).
   (b) \( 6\pi \)  \( \theta = 6\pi \)  \( \theta_R = \frac{3\pi}{2} \).
   (c) \( -7\pi \)  \( \theta = -7\pi \)  \( \theta_R = -\frac{7\pi}{2} \).
   (d) \( \frac{5\pi}{2} \)  \( \theta = \frac{5\pi}{2} \)  \( \theta_R = \frac{\pi}{2} \).
   (e) \( \frac{7\pi}{2} \)  \( \theta = \frac{7\pi}{2} \)  \( \theta_R = \frac{\pi}{2} \).

In Problems 3 to 14, replace each angle by a nonnegative coterminal angle that is less than one revolution and then find the exact values of the six trigonometric functions (if they are defined) for the angle.

3. \( 1440^\circ \)
4. \( 810^\circ \)
5. \( 900^\circ \)
6. \( -220^\circ \)
7. \( 750^\circ \)
8. \( 1845^\circ \)
9. \( -675^\circ \)
10. \( \frac{19\pi}{2} \)
11. \( 5\pi \)
12. \( \frac{25\pi}{6} \)
13. \( \frac{17\pi}{3} \)
14. \( -\frac{31\pi}{4} \)
15. What happens when you try to evaluate $\tan 90^\circ$ on a calculator? [Try it.]

16. Let $\theta$ be a quadrant III angle in standard position and let $\theta_R$ be its reference angle. Show that the value of any trigonometric function of $\theta$ is the same as the value of $\theta_R$, except possibly for a change of algebraic sign. Repeat for $\theta$ in quadrant IV.

In problems 17 to 36, find the reference angle $\theta_R$ for each angle $\theta$, and then find the exact values of the six trigonometric functions of $\theta$.

17. $\theta = 150^\circ$  
18. $\theta = 120^\circ$  
19. $\theta = 240^\circ$  
20. $\theta = 225^\circ$  
21. $\theta = 315^\circ$  
22. $\theta = 675^\circ$  
23. $\theta = -150^\circ$  
24. $\theta = -\frac{5\pi}{6}$  
25. $\theta = -60^\circ$  
26. $\theta = -\frac{13\pi}{6}$  
27. $\theta = -\frac{\pi}{4}$  
28. $\theta = \frac{53\pi}{6}$  
29. $\theta = -\frac{2\pi}{3}$  
30. $\theta = \frac{9\pi}{4}$  
31. $\theta = \frac{7\pi}{4}$  
32. $\theta = -\frac{50\pi}{3}$  
33. $\theta = \frac{11\pi}{3}$  
34. $\theta = -\frac{147\pi}{4}$  
35. $\theta = -420^\circ$  
36. $\theta = -5370^\circ$

37. Complete the following tables. (Do not use a calculator.)

<table>
<thead>
<tr>
<th>$\theta$ degrees</th>
<th>$\theta$ radians</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>210°</td>
<td>$\frac{7\pi}{6}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>225°</td>
<td>$\frac{5\pi}{4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>240°</td>
<td>$\frac{4\pi}{3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300°</td>
<td>$\frac{5\pi}{3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>315°</td>
<td>$\frac{7\pi}{4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>330°</td>
<td>$\frac{11\pi}{6}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta$ degrees</td>
<td>$\theta$ radians</td>
<td>$\cot \theta$</td>
<td>$\sec \theta$</td>
<td>$\csc \theta$</td>
</tr>
<tr>
<td>------------------</td>
<td>-----------------</td>
<td>---------------</td>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td>210°</td>
<td>$\frac{7\pi}{6}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>225°</td>
<td>$\frac{5\pi}{4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>240°</td>
<td>$\frac{4\pi}{3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300°</td>
<td>$\frac{5\pi}{3}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>315°</td>
<td>$\frac{7\pi}{4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>330°</td>
<td>$\frac{11\pi}{6}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

38. Use a calculator to complete the tables to investigate the behavior of $\tan \theta$ as $\theta$ approaches $90^\circ$. Round each value to the nearest integer.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$89^\circ$</th>
<th>$89.9^\circ$</th>
<th>$89.99^\circ$</th>
<th>$89.999^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta$</td>
<td>$91^\circ$</td>
<td>$90.1^\circ$</td>
<td>$90.01^\circ$</td>
<td>$90.001^\circ$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In problems 39 to 42, use a calculator to verify that the equation is true for the indicated value of the angle $\theta$.

39. $\tan \theta = \frac{\sin \theta}{\cos \theta}$ for $\theta = 35^\circ$.

40. $(\cos \theta)(\tan \theta) = \sin \theta$ for $\theta = \frac{5\pi}{7}$

41. $\cos^2 \theta + \sin^2 \theta = 1$ for $\theta = \frac{5\pi}{3}$

42. $1 + \tan^2 \theta = \sec^2 \theta$ for $\theta = 17.75^\circ$

43. Verify that for $\theta = 0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, and $90^\circ$,
we have $\sin \theta = \frac{\sqrt{0}}{2}$, $\frac{\sqrt{1}}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{4}}{2}$ respectively.

[Although there is no theoretical significance to this pattern, people often use it as a memory aid to help recall these values of $\sin \theta$.]