# Lectures in Plasma Physics

This manuscript is the written version of seven two-hour lectures that I gave as a minicourse in Plasma Physics at Illinois Institute of Technology in 1978 and 1980. The course, which was open to all IIT students, served to introduce the basic concept of plasmas and to describe the potentially practical uses of plasma in communication, generation and conversion of energy, and propulsion. A final exam, which involves nomenclature and concepts in plasma physics, is included at the end.

It is increasingly evident that the plasma concept plays a basic role in current research and development. However, it is not particularly simple for a science or engineering student to obtain a qualitative understanding of the basic properties of plasmas. In a plasma, one must understand and apply the principles of electromagnetism (and electromagnetic waves in a non-linear diamagnetic medium!) dynamics, thermodynamics, fluid mechanics, statistical mechanics, chemical equilibrium, and atomic collision theory. Furthermore, in plasma one quite frequently encounters non-linear, collective effects, which cannot properly be analyzed without sophisticated mathematical techniques. A student typically has not completed the prequisites for a basic course in plasma physics until and unless s/he enters graduate school.

For these lectures I have tried to put across the basic physical concepts, without bringing in the elegant formalism, which is irrelevant in this introduction. In giving these lectures, I have assumed that the student has had course work at the level of Physics: Part I and Physics: Part II by Resnick and Halliday, so that s/he is acquainted with the basic concepts of mechanics, wave motion, thermodynamics, and electromagnetism. For students of science and engineering who had this sophomore-level physics, I hope that these lectures serve as a proper introduction to this stimulating, rapidly developing and perhaps crucially important field.

I have used SI (MKS) units for electromagnetic fields, rather than the Gaussian system that has been more conventional in plasma physics, in order to make the subject more accessible to undergraduate science and engineering students.

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# Chapter 1

# Introduction to Plasmas

## 1.1 Preliminary

In the ancient phlogiston theory there was a classification of the states of matter: i. e., “earth”, “water”, “ air”, and “fire”. While the phlogiston theory had certain basic defects, it did properly enumerate the four states of matter – solid, liquid, gas, and plasma. It is estimated that more than 99% of the matter in the universe exists in a plasma state; however, the significance and character of the plasma state has been recognized only in the twentieth century. The states of solid, liquid, gas, and plasma represent correspondingly increasing freedom of particle motion. In a solid, the atoms are arranged in a periodic crystal lattice; they are not free to move, and the solid maintains its size and shape. In a liquid the atoms are free to move, but because of stong interatomic forces the volume of the liquid (but not its shape) remains unchanged. In a gas the atoms move freely, experiencing occasional collisions with one another. In a plasma, the atoms are ionized and there are free electrons moving about – a plasma is an ionized gas of free particles.

Here are some familiar examples of plasmas:

1. Lightning, Aurora Borealis, and electrical sparks. All these examples show that when an electric current is passed through plasma, the plasma emits light (electromagnetic radiation).

2. Neon and fluorescent lights, etc. Electric discharge in plasma provides a rather efficient means of converting electrical energy into light.

3. Flame. The burning gas is weakly ionized. The characteristic yellow color of a wood flame is produced by 579 $nm$ transitions (D lines) of sodium ions.

4. Nebulae, interstellar gases, the solar wind, the earth's ionosphere, the Van Allen belts. These provide examples of a diffuse, low temperature, ionized gas.

5. The sun and the stars. Controlled thermonuclear fusion in a hot, dense plasma provides us with energy (and entropy!) on earth. Can we develop a practical scheme for tapping this virtually inexhaustible source of energy?

The renaissance in interest and enthusiasm in plasma physics in recent years has occurred because, after decades of difficult developmental work, we are on the verge of doing "break - even" experiments on energy production by controlled fusion. We still have a long way to go from demonstrating scientific feasibility to developing practical power generating facilities. The current optimism among plasma physicists has come about because there does not seem to be any matter of principle that could (somewhat maliciously!) prevent controlled thermonuclear fusion in laboratory plasmas. The development of this energy source will require much insight and ingenuity – like the development of sources of electrical energy.[1]

Although plasma physics may well hold the key to virtually limitless sources of energy, there are a number of other applications or potential applications of plasma, such as the following:

1. Plasma may be kept in confinement and heated by magnetic and electric fields. These basic features of plasmas could be used to build "plasma guns" that eject ions at velocities up to 100 km/sec. Plasma guns could be used in ion rocket engines, as an example.

2. One could build "plasma motors", which differ from ordinary motors by having plasma (not metals!) as the basic conductor of electricity. These motors could, in principle, be lighter and more efficient than ordinary motors. Similarly, one could develop "plasma generators" to convert mechanical energy into electical energy. The whole subject of direct magnetohydrodynamic production of electical energy is ripe for development. The current "thermodynamic energy converters" (steam plants, turbines, etc.) are notoriously inefficient in generating electricity.

3. Plasma could be used as a resonator or a waveguide, much like hollow metallic cavities, for electromagnetic radiation. Plasma experiences a whole

[1] Napoleon Bonaparte, who was no slouch as an engineer, felt that it would never be practical to transmit electric current over a distance as large as one kilometer.

range of electrostatic and electromagnetic oscillations, which one would be able to put to good use.

4. Communication through and with plasma. The earth's ionosphere reflects low frequency electromagnetic waves (below 1 MHz) and freely transmits high frequency (above 100 MHz) waves. Plasma disturbances (such as solar flares or communication blackouts during re-entry of satellites) are notorious for producing interruptions in communications. Plasma devices have essentially the same uses in communications, in principle, as semi-conductor devices. In fact, the electrons and holes in a semi-conductor constitute a plasma, in a very real sense.

## 1.2 Definition and Characterization of the Plasma State

A plasma is a quasi-neutral "gas" of charged and neutral particles that exhibits collective behavior. We shall say more about collective behavior (shielding, oscillations, etc.) presently. The table given below indicates typical values of the density and temperature for various types of plasmas. We see from the table that the density of plasma varies by a factor of $10^{16}$ and the temperature by $10^4$ – this incredible variation is much greater than is possible for the solid, liquid, or gaseous states.

| Plasma Type | Particle Density $\#/cm^3$ | Temp. $T$ (K) | Debye Length (cm) | Plasma Freq. $\omega_P$ (Hz) | Collision Freq. $\nu$ (Hz) |
|---|---|---|---|---|---|
| Interstellar Gas | 1 | $10^4$ | $10^3$ | $6\times10^4$ | $10^{-4}$ |
| Solar Corona | $10^6$ | $10^6$ | 1 | $6\times10^7$ | $10^{-1}$ |
| Solar Atm. Gas Discharge | $10^{14}$ | $10^4$ | $10^{-4}$ | $6\times10^{11}$ | $3\times10^9$ |
| Diffuse Lab Plasma | $10^{12}$ | $10^6$ | $10^{-2}$ | $6\times10^{10}$ | $10^5$ |
| Dense Lab Plasma | $10^{14}$ | $10^6$ | $10^{-4}$ | $6\times10^{12}$ | $5\times10^8$ |
| Thermonuclear Plasma | $10^{16}$ | $10^8$ | $10^{-3}$ | $6\times10^{12}$ | $8\times10^5$ |

Although plasma does occur under a variety of conditions, it seldom occurs too close to our environment because a gas in equilibrium at STP, say, has essentially no ions. The energy levels of a typical atom are shown.

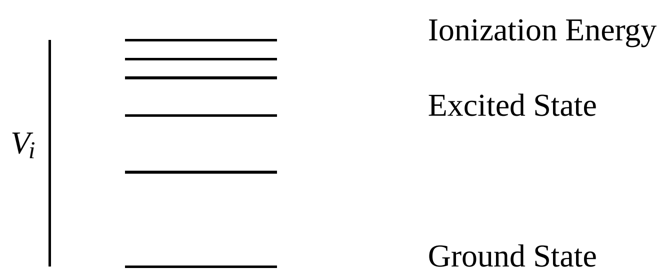


Figure 1.1: Energy Levels of a Typical Atom

The ionization potential, $V_i$, is the energy necessary to ionize an atom. Typically, $V_i$ is 10 electron Volts or so. It is relatively improbable for atoms to be ionized, unless they have a kinetic energy that is comparable in magnitude to this ionization potential. The average kinetic energy of atoms in a gas is of order $k_BT$, where $k_B$ is Boltzmann's constant and $T$ is the (Kelvin) temperature. At 300 K, the factor $k_BT$ is about $1/40$ eV, whereas it is 1 eV at 11600 K. In plasma physics, it is conventional to express the temperature by giving $k_BT$ in electron Volts. Note: $1\,eV = 1.6\times10^{-19}$ Joules.

The equilibrium of the ionization reaction of an atom $A$,

$$A \rightarrow A^+ + e^-$$

may be treated by the law of chemical equilibrium. Let $n_e$ be the number of electrons per unit volume, $n_i$ be the number of ions per unit volume ($n_e = n_i$ for a neutral plasma), and $n_n$ the number of neutral particles per unit volume. The ratio

$$\frac{n_i n_i}{n_n} = \overline{K}(T) \tag{1.1}$$

gives the thermodynamic rate constant $\overline{K}(T)$. One may compute the quantity $\overline{K}$ by detailed quantum mechanical analysis. The result is

$$\overline{K}(T) = \frac{a}{\lambda^3} \mathrm{e}^{-V_i/(k_B T)} \tag{1.2}$$

where $\lambda$ is the average DeBroglie wavelength of ions at temperature $T$, and $a$ is an "orientation factor" which is of order 1. As a rough estimate,

$$\lambda \approx h/\sqrt{M k_b T}$$

for ions of mass $M$, where $h$ is Planck's constant.

It is evident from the Table that plasmas exist either at low densities, or else at high temperatures. At low densities the left side of Eq. (1) is small, even if the plasma is fully ionized, and even if the temperature (and consequently the rate constant) is low. It takes a long time for such a plasma to come to equilibrium because the time between collisions is long. However, the probability of recombination collisions is very small, and the diffuse gas remains ionized. By contrast, at high temperatures ($10^6$ K or larger) the rate constant is large, and ionization occurs, even in a dense plasma.

We shall discuss "quasi-neutral" plasmas, which have equal numbers of electrons and ions. The positive and negative particles move freely in a plasma. As a consequence, electric fields are neutralized in a plasma, just as they are in a metalic conductor. Free charges are "shielded out" by the plasma, as shown in the diagram.

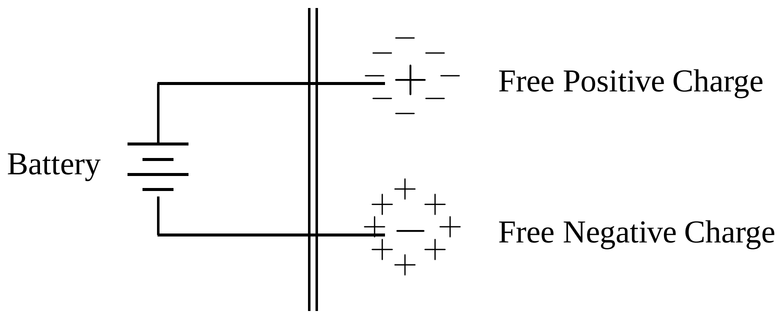


Figure 1.2: Shielding by Plasma

However, in the region very close to the free charge, it is not shielded effectively and electric fields may exist. The characteristic distance for shielding in a plasma is the Debye length $\lambda_D$.

$$\lambda_D = \sqrt{\frac{k_B T \varepsilon_0}{n e^2}} \tag{1.3}$$

with $n$ charge carriers of charge $e$ per unit volume. In practical units

$$\lambda_D = 740\sqrt{k_B T / n}\,\mathrm{cm}$$

with $k_B T$ in electron Volts and $n$ as the number per cm$^3$.

I shall discuss shielding in a case that is simple to analyze: an infinite sheet of surface charge density $\sigma$ is placed inside a plasma. We choose a Gaussian surface of cross-sectional area $A$ and length $x$, as shown.

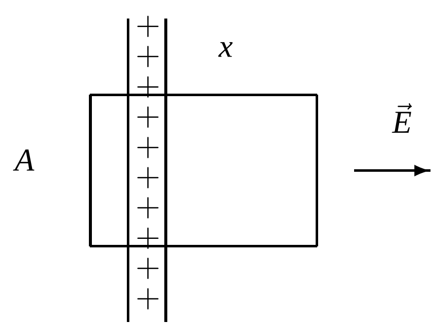


Figure 1.3: Gaussian Surface

We apply Gauss's law,

$$\varepsilon_0 \oint \vec{E} \cdot \vec{dS} = q_{enc} \tag{1.4}$$

to this surface to obtain

$$\varepsilon_0 A \left(E_x(x) - E_x(0)\right) = q_{enc} = A\left(\sigma + \int_0^x (n_i - n_e) e dx\right) \tag{1.5}$$

Differentiating, we get

$$\varepsilon_0 \frac{dE_x}{dx} = e(n_i - n_e) \tag{1.6}$$

By symmetry, the electric field has only an $x$-component, which for electric potential $V(x)$ is given by $E_x = -dV/dx$, so that

$$\frac{d^2V}{dx^2} = -\frac{e}{\varepsilon_0}(n_i - n_e) \tag{1.7}$$

In the plasma the more massive ions and neutral atoms are relatively immobile in comparison with the lighter electrons. The electrons populate the region of high potential according to the Boltzmann formula

$$n_e(x) = n_\infty \exp\left[\frac{eV(x)}{k_B T}\right] \tag{1.8}$$

where $n_i = n_\infty$ is the density of electrons and ions far away. At low potential ($eV(x) \ll k_B T$) one has

$$n_i - n_e = n_\infty \left(1 - \exp\left[\frac{eV(x)}{k_B T}\right]\right) \approx -\left[\frac{e\, n_\infty}{k_B T}\right] V(x) \tag{1.9}$$

Thus

$$\frac{d^2V}{dx^2} = \frac{e^2\, n_\infty}{k_B T\, \varepsilon_0} V(x) = \frac{1}{\lambda_D^2} V(x) \tag{1.10}$$

The potential in this case is

$$V(x) = E_0 \lambda e^{-x/\lambda_D} \tag{1.11}$$

The electric field, $E(x) = E_0 \exp[-x/\lambda_D]$, is exponentially shielded by the plasma, with characteristic distance $\lambda_D$.

The Debye length $\lambda_D$ is very small except for the most tenuous of plasmas. We require that it be much smaller than the size of the plasma. We shall also require that the potential energy of the electrons always be much smaller than $k_B T$, a typical electronic kinetic energy. This latter requirement is equivalent to the condition $n\lambda_D^3 \gg 1$. In other words, we require that many particles lie within the shielding volume.

Finally, I shall talk about plasma oscillations. We have just seen that static electric fields are shielded out in a plasma. If a positive charge is suddenly brought into a plasma, the negative charges nearby begin to move toward that charge. Because of their inertia, they "overshoot", creating a net negative charge imbalance. The electrons then move away from that region; and so forth. The characteristic frequency of these oscillations is the plasma frequency $\omega_P$:

$$\omega_P = \sqrt{\frac{ne^2}{m\varepsilon_0}} \tag{1.12}$$

In practical units,

$$\omega_P = 5 \times 10^4 \sqrt{n}\,\mathrm{Hz}$$

I shall demonstrate this plasma oscillation by assuming that all of the electrons are somehow removed from an infinite slice of plasma of thickness $x$, and that twice as many electronc as usual exist in a nearby parallel slice of thickness $x$, as shown in the figure.

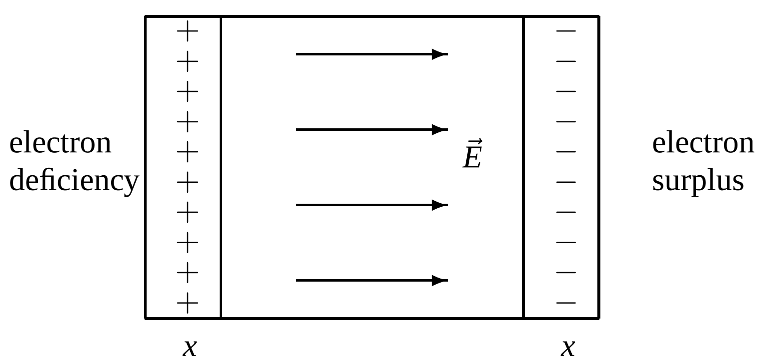


Figure 1.4: Charged Slabs

An electric field is set up between the two plates, as shown. The net (positive) charge per unit area in the electron-deficient slice is $\sigma = nex$. From Gauss's law,

$$E = \frac{nex}{\varepsilon_0} \tag{1.13}$$

The electric field acts on the electrons in the intermediate region, causing them to move to the left, and causing the slice thickness to decrease. (The acceleration of each electron is $\ddot{x}$, in fact.) Thus, we obtain from Newton's second law that

$$m\ddot{x} = -eE = -\frac{ne^2}{\varepsilon_0}x \tag{1.14}$$

Thus, $x(t) = x_0 \cos(\omega_P t + \delta)$, so that the charge imbalance oscillates with (angular) frequency $\omega_P$.

The particles in a plasma make collisions with one another – the more dense the plasma, the more frequent the collisions. Let $\nu$ be the collision frequency, and $\tau = 1/\nu$ the time between collisions. For the plasma state we require $\omega_P \tau \gg 1$. As a consequence, it is improbable for the electrons to collide during a single plasma oscillation. Thus, the plasma oscillations are not merely damped out because of interparticle collisions. The electrons undergo these oscillations without any real collisional damping, and such oscillations are characteristic of the plasma state. You will hear more about them.

A magnetic field plays an absolutely basic role in plasma physics; it penetrates the plasma, even though the electric field does not. I shall briefly review the motion of a charged particle in a uniform $\vec{B}$ field. The particle experiences the Lorentz force,

$$\vec{F} = q\vec{v} \times \vec{B}$$

By Newton's second law, a charged particle moves with constant speed in the direction parallel to $\vec{B}$, and moves in a uniform circular orbit in the plane perpendicular to $\vec{B}$. If $v_\perp$ is the particle speed in the circle and $r_\perp$ is the radius of the circular orbit, then

$$qv_\perp B = \text{force} = \text{mass} \times \text{acceleration} = \frac{mv_\perp^2}{r_\perp} \tag{1.15}$$

The particle moves in a circular orbit with cyclotron frequency

$$\omega = v_\perp / r_\perp = qB/m$$

The radius of the circular orbit, Larmor radius

$$r_\perp = mv_\perp/(qB)$$

depends on the initial speed of the particle in the plane perpendicular to $\vec{B}$. The composite motion of the charged particle is a helical spiral motion along one of the lines of magnetic induction. Oppositely charged particles spiral around the field line in the opposite sense.

# Chapter 2

# Single Particle Dynamics

## 2.1 Introduction

This lecture deals exclusively with the motions of a single charged particle in external fields, which may be electrical, magnetic, or gravitational in character. The motions of charged particles in external fields are the full story in cyclotrons, synchrotrons, and linear accelerators – the mving charges themselves produce negligibly small fields. The charged particle beams in such devices aren't plasmas, since collective effects don't matter. In fact, nobody knows how to build high-current accelerators in which collective (space charge) effects are large – this is a very costly limitation of accelerator technology. In a plasma, collective effects occur and are likely to be important. For some purposes, we may treat plasma as a collection of individual particles, and sometimes we must handle the plasma as a continuous fluid.

The Lorentz force, $\vec{F} = q\vec{v} \times \vec{B}$, causes a charged particle (charge $q$, mass $m$) to spiral around a line of force in a uniform magnetic field. In the plane perpendicular to the field, the Larmor radius $r_\perp$ is determined by the initial transverse speed $v_\perp$ by the relation

$$r_\perp = \frac{mv_\perp}{qB} = \frac{\sqrt{2m\mathcal{E}\perp}}{qB} \tag{2.1}$$

with the transverse kinetic energy $\mathcal{E}_\perp = mv_\perp^2/2$. In practical units ($B$ in Gauss, $\mathcal{E}_\perp$ in electron Volts),

$$r_\perp = 3.37\sqrt{\mathcal{E}_\perp}/B\,\text{cm}$$

## 2.2 Orbit Theory

In the branch of accelerator (plasma) physics known as orbit theory, one assumes that there is present a magnetic field $\vec{B}$ that is approximately uniform over a particle orbit, so that, to a first approximation, the charged particles spiral around field lines. The charged particles are, however, slightly affected by one or more of the following small perturbative contributions:

- A small (time-independent) electric or gravitational field.
- A slight non-uniformity in the magnetic field, with its magnitude ($|\vec{B}|$) changing with position).
- A slight curvature of the lines of force of $\vec{B}$.

## 2.3 Drift of Guiding Center

As a consequence lf any of these small sources of perturbation, the guiding centers of the particle cyclotron orbits experience drifts. These guiding center drifts are not negligible. even though they come from small perturbations, because they often have a large cumulative effect over a long time. It is an excellent approximation to think in terms of guiding center drifts with a distance $\ell$, over which non-uniformities or small perturbations are important, as very large in comparison with the Larmor radius $r_\perp$; ($\ell \gg r_\perp$).

We need consider only forces in the plane perpendicular to the uniform $\vec{B}$ field – those parallel to $\vec{B}$ simply accelerate the particle along $\vec{B}$, and do not affect the transverse motion. Also, note that in a pure magnetic field – be it non-uniform, curved, or whatever – the Lorentz force does no work on the particle:

$$\vec{v} \cdot \vec{F} = q\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$$

Consequently, the mechanical energy of the particle does not change.

The simplest case to illustrate guiding center drift is that of a constant force $\vec{F}$ in plane perpendicular to a uniform magnetic induction $\vec{B}$. Let the force act in the $x$-direction, with the $\vec{B}$ field in the $z$-direction. The mechanical energy of the system, $\frac{1}{2}mv_\perp^2 - Fx$, is a constant of the motion. As a consequence, the presence of the force $\vec{F}$ perturbs the circular Larmor orbit; there are greater speeds, and consequently larger radii of curvature (see Eq.(1)) for $x > 0$ than for $x < 0$,

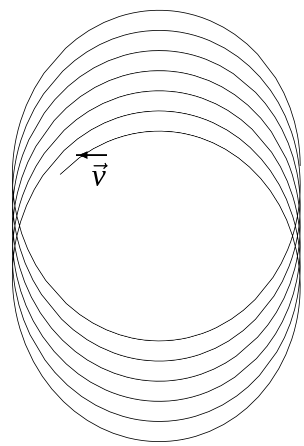


Figure 2.1: Drift of Guiding Center

As a consequence, for counterclockwise circulation there is an upward drift of the guiding center for a positive charge, as shown in Figure 2.1.

We can analyze this motion quantitatively using Newton's second law:

$$m\frac{d\vec{v}}{dt} = \vec{F} + q\vec{v} \times \vec{B} \tag{2.2}$$

for the *x*- and *y*-components:

$$\begin{aligned} m\frac{dv_x}{dt} &= F + qv_yB \\ m\frac{dv_y}{dt} &= -qv_xB \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{qB}{m}\left(v_y + \frac{F}{qB}\right) \\ \frac{d}{dt}\left(v_y + \frac{F}{qB}\right) = \frac{dv_y}{dt} &= -\frac{qB}{m}v_x \end{aligned} \tag{2.4}$$

If we define a velocity $\vec{u} = \vec{v} + F/(qB)\hat{y}$, Eqn. (4) represents uniform circular motion with angular velocity $\omega = qB/m$ for the velocity vector $\vec{u}$. As a consequence, the composite motion is circular motion with angular velocity $\omega$, with respect to a guiding center that is drifting downward with speed $F/(qB)$. Consequently, the drift velocity of the guiding center is

$$\vec{v}_D = -\hat{j}\frac{F}{qB} = \frac{1}{qB^2}\vec{F}\times\vec{B} \tag{2.5}$$

If the force $\vec{F}$ is produced by a uniform electric field $\vec{E}$, then $\vec{F} = q\vec{E}$ and

$$\vec{v}_D = \frac{\vec{E}\times\vec{B}}{B^2} \tag{2.6}$$

Thus, in a uniform electric field, all charged particles experience an $\vec{E}\times\vec{B}$ drift, which is independent of the magnitude or sign of their charge, or their mass.

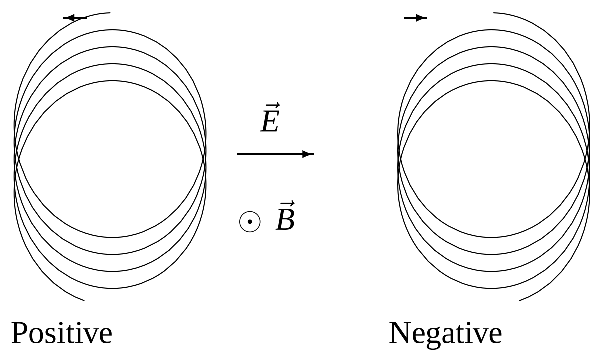


Figure 2.3: Drift of Positive and Negative Charge

We shall discuss several other circumstances in which there are guiding center drifts in response to small perturbations to a uniform $\vec{B}$ field. There is a systematic procedure for analyzing such perturbations, which consists of (i) computing the average residual force on a charged particle as it moves across a circular Larmor orbit, and (ii) using Eq.(5) to compute the drift speed. We illustrate this procedure in discussing Gradient $\vec{B}$ Drift.

Let us assume that the $\vec{B}$ field has fixed direction, with the magnitude increasing as one moves transverse to $\vec{B}$, to the left as shown in Figure 2.4. The Larmor radius, $r_\perp = mv_\perp/(qB)$, varies inversely to $B$. Thus, the left half of the orbit should have a larger radius than the right half, as shown. Positive charges should drift upward and negative charges should drift downward, as shown in Figure 2.4.

We shall compute the drift speed for positive charges. The magnetic induction is assumed to vary linearly with $x$:

$$\vec{B} = \left[B_0 + x\frac{dB}{dx}\right]\hat{z} \tag{2.7}$$

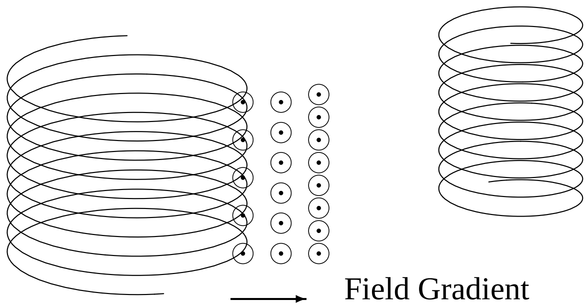


Figure 2.4: Drift in Gradient $\vec{B}$ Field

where $B_0$ is the magnitude of $\vec{B}$ at $x = 0$, and $(dB/dx)_0$ is its gradient at $x = 0$. We neglect any quadratic variation in $x$, by assuming the Larmor radius is relatively small. The Lorentz force on a particle in a circular orbit is

$$\vec{F} = q\vec{v} \times \vec{B} = -qvB\hat{r} \tag{2.8}$$

The force is radially inward, as shown.

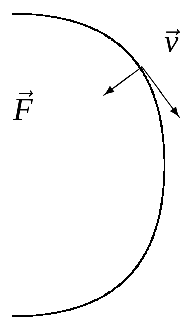


Figure 2.5: Circular Orbit

The Lorentz force may be written as

$$\vec{F} = -qvB_0\hat{r} - qv\frac{dB}{dx}x\hat{r} \tag{2.9}$$

the first term being responsible for the circular motion, and the second being the residual force. The average residual force over one cycle is

$$\vec{F}_{residual} = -\frac{qv}{r}\frac{dB}{dx}\left[\langle x^2\rangle\hat{i} + \langle xy\rangle\hat{j}\right] = -\frac{qv}{2}\frac{dB}{dx}r_\perp\hat{i} \tag{2.10}$$

since by symmetry $\langle xy \rangle = 0$ and $\langle x^2 \rangle = \frac{1}{2} r_\perp^2$. We use (2.10) in Eq. (2.5) to obtain the drift speed

$$\vec{v}_D = \frac{qV}{2} \frac{dB}{dx} \hat{j} \tag{2.11}$$

In general

$$\vec{v}_D = \pm \frac{q v_\perp}{2B^2} (\vec{B} \times \mathrm{grad} \vec{B}) \tag{2.12}$$

with the $\pm$ signs taken for positive (negative) particles.

An additional source of charged particle drift is the curvature of magnetic field lines. We shall discuss the case in which the magnetic field is of constant magnitude, with the field lines curved. A charged particle spirals around a particular field line, as shown:

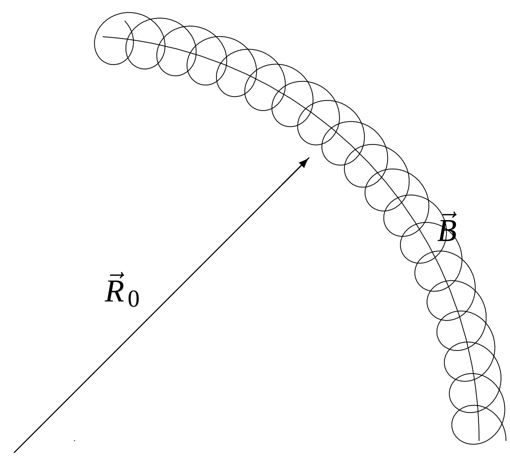


Figure 2.6: Spiral around Field Line

If the radius of curvature of the field line (approximately circular) is $\vec{R}_0$, the guiding center follows a spiral around the field line, as shown. The motion of the charged particle is most conveniently analyzed in a non-inertial coordinate system with the guiding center at the origin. In that system, the charged particle experiences an inertial centrifugal force

$$\vec{F} = \frac{m v_\parallel}{R_0^2} \vec{R}_0 \tag{2.13}$$

with $v_\parallel$ the speed parallel to the field lines. Thus, the curvature drift velocity comes out to be

$$\vec{v}_D = \frac{1}{qB^2} \vec{F} \times \vec{B} = \frac{m v_\parallel^2}{qB^2} \vec{R}_0 \times \vec{B} \tag{2.14}$$

Remark: It is impossible to draw field lines for a $\vec{B}$-field that are curved, without having $|\vec{B}|$ to change, as well. Consequently, field gradients are always present whenever the magnetic field lines are curved; we discuss them separately for the sake of simplicity. As an example, let us consider a simple toroidal field. If a total current is distributed uniformly and without "twist" on the surface of the torus, the magnetic induction has only a polar component

$$B_0 = \frac{\mu_0 I}{2\pi R_0} \tag{2.15}$$

where $R_0$ is the radius of curvature of the field line, The field gradient is

$$\text{grad}\,|\vec{B}| = -\frac{\mu_0 I}{2\pi}\frac{\vec{R}_0}{R_0^2}$$

and the grad$B$ drift is

$$\vec{V}_D = \pm\frac{v_\perp r_\perp}{2}\frac{\vec{R}_0 \times \vec{B}}{R_0^2 B} = \frac{m v_\perp^2}{2qB}\frac{\vec{R}_0 \times \vec{B}}{R_0^2 B} \tag{2.16}$$

The overall drift for this toroidal field is

$$\vec{v}_D = \frac{m}{q}\frac{\vec{R}_0 \times \vec{B}}{R_0^2 B^2}\left(V^2 + v_\perp^2/2\right) \tag{2.17}$$

## 2.4 Adiabatic Drift

A particle in a uniform magnetic field spirals along a field line. If the field is slightly non-uniform, the particle either slowly drifts or spirals into regions of space that have different magnetic fields. Under such drifts, the radius of gyration of the orbit changes sufficiently slowly, and there are certain adiabatic invariants associated with the orbit, which do not change. One such adiabatic invariant is the angular momentum action.

$$\oint L_\parallel \, d\theta = 2\pi L_\parallel$$

the integral of the component of angular momentum about $\vec{B}$, taken over one gyration. Thus, the angular momentum component taken about the guiding center, and parallel to the field, does not change.

$$\begin{aligned} L_\parallel &= m v_\perp r_\perp = \frac{m v_\perp^2}{\omega} = m v_\perp^2 \frac{m}{eB} = \frac{m^2 v_\perp^2}{eB} \\ &= \frac{2m}{e}\frac{\mathcal{E}_\perp}{B} \end{aligned} \tag{2.18}$$

The quantity $\mathcal{E}_\perp = mv_\perp^2/2$, the transverse kinetic energy, is the contribution to the kinetic energy from transverse motion. From (2.18), we see that as the particle spirals or drifts into a region of increasing magnetic field, the transverse speed $v_\perp$ and energy $E_\perp$ both increase.

As a particle of charge $e$ moves with angular frequency $\omega$ in a circular orbit of radius $r_\perp$, one may define an average current $I = e/T = e\omega/(2\pi)$, and a magnetic moment

$$\begin{aligned} \mu &= \text{current} \times \text{area} = \frac{e\omega}{2\pi}\pi r_\perp^2 \\ &= \frac{e\omega}{2} r_\perp^2 = \frac{e v_\perp^2}{2\omega} = \frac{e v_\perp^2}{2}\frac{m}{eB} = \frac{\mathcal{E}_\perp}{B} \end{aligned} \tag{2.19}$$

Consequently, the magnetic moment of the "current loop" is also an adiabatic invarian. The flux of magnetic field through the loop, $\pi r_\perp^2 B$, is an equivalent adiabatic invariant. The Larmor radius $r_\perp$ does decrease when the magnetic field increases, since

$$r_\perp = \frac{mv_\perp}{eB} = \frac{2}{e}\frac{\mu}{v_\perp} \tag{2.20}$$

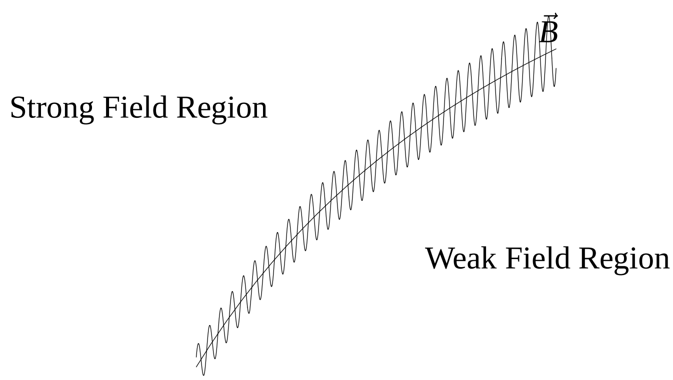


Figure 2.7: Spiral Curve with $|\vec{B}|$ Changing Slightly

As a particle spirals from a weak field to a strong field region, its perpendicular speed $v_\perp$ increases. However, since its total kinetic energy

$$\mathcal{E} = \frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right)$$

must remain fixed, the longitudinal speed $v_{||}$ must decrease. If the particle spirals along a field line to a point at which $v_{||}$ is zero, the particle is reflected.

## 2.5 Magnetic Mirror

These features form the basis for a plasma confinement system known as a Magnetic Mirror. (See Figure 2.8.) Plasma is trapped in the central weak field region, being reflected as it enters the strong field region near the coils.

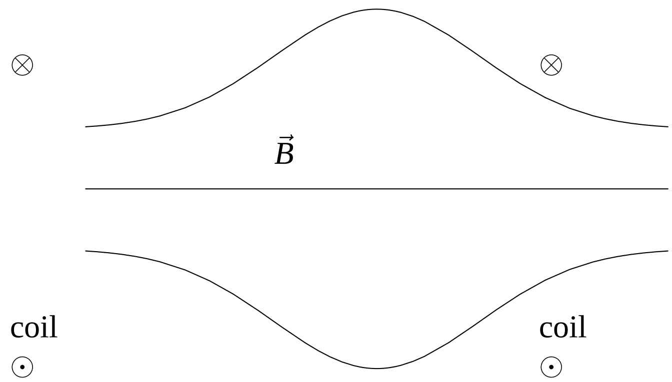


Figure 2.8: Magnetic Mirror

The magnetic mirror is not a perfect confining device for plasma, since if $v_{\perp}$ is very small to begin with, it remains small and thus $v_{||}$ never gets small. The net result is escape for that charged particle. In fact, if $v_{\perp}$ has the value $v_{0\perp}$ has the value in the weak field region, its value in the strong field region is

$$v_{\perp\, max} = \frac{B_{max}}{B_0} v_{0\perp} \tag{2.21}$$

If plasma goes past the point of maximum field, $v_{||} \geq 0$ at this point, and

$$v^2_{\perp\, max} \leq v^2_{0||} + v^2_{0\perp} \tag{2.22}$$

We thus obtain

$$1 + \left(\frac{v_{0||}}{v_{0\perp}}\right)^2 \geq \frac{B_{max}}{B_0} \tag{2.23}$$

as a condition for escape of plasma.

If we define the mirror ratio

$$\frac{B_{max}}{B_0} = \sin^2\theta \tag{2.24}$$

the condition for escape may be written as

$$\frac{v_{0\perp}}{v_{0\parallel}} \leq \tan\theta \tag{2.25}$$

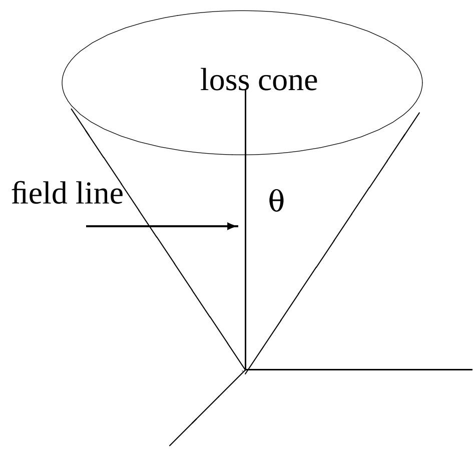


Figure 2.9: Loss Cone

In other words, all particles within the loss cone escape. As a consequence, the mirror-confined plasma is definitely non-isotropic.

Electrons and ions are equally well confined in a magnetic mirror, although electrons are more likely to escape through collisions. Since it is hard for particles to escape from a magnetic mirror, it is equally hard to get them inside the magnetic mirror in the first place. Usually they are put inside the bottle with an injector. If their motion were perfectly adiabatic, they would come back to hit the injector, so it is just as well for their motion to be slightly non-adiabatic!

As a charged particle spirals along a field line, and is reflected at $a$ and $b$ by the strong fields, it in general experiences transverse drifts because of external fields, etc. An adiabatic invariant for these drifts is the “longitudinal invariant”,

$$J = \int_a^b v_\parallel \, ds \tag{2.26}$$

where $s$ is the path length along the guiding center that lies along a field line. If drifts occur over a time scale that is long in comparison with the period of oscillation from $a$ to $b$, then $J$ remains an adiabatic invariant.

We may apply these concepts of drift, adiabatic invariants and such to the motion of charged particles in the earth's magnetic field. The earth is like a giant bar magnet with magnetic dipole moment $M = 8 \times 10^{25}$ Gauss cm$^3$. (The North pole of the earth is closest to the south pole of the magnet.) At the equator on the earth's surface ($R = 6.4 \times 10^6$ m).

$$B \approx \frac{M}{R^3} = 0.2\,\text{Gauss} \tag{2.27}$$

As one goes along a field line toward the poles, the magnetic field increases in magnitude. Particles in the Van Allen belts are trapped in the earth's magnetic field; they execute latitude oscillations about the equator.[1] Their motion is simple harmonic, with angular frequency

$$\omega_{lat} = \frac{3\omega}{\sqrt{2}}\frac{r_\perp}{R} \tag{2.28}$$

where $r_\perp$ is the Larmor radius and $\omega$ the cyclotron frequency at the equation; $R$ is the distance to the center of the earth.

There is also a longitude drift because of field gradients:

$$v_{grad} = \frac{\omega r_\perp^2}{2B} \cdot \text{grad}\,B = \frac{3\omega r_\perp^2}{2R} \tag{2.29}$$

The angular frequency of longitude drift is

$$\omega_{long} = \frac{v_{grad}}{R} = \frac{3\omega}{2}\frac{R^2}{r_\perp^2} \tag{2.30}$$

For 10 keV electrons at $R = 3 \times 10^7$ m (five earth radii) and $r_\perp = 10^3$ m, the period of latitude oscillation is about 1.4 sec, and the longitude period (time necessary to drift around the earth) is $7 \times 10^4$ sec. The earth's magnetic field is $3 \times 10^{-3}$ Gauss, and the cyclotron frequency is $\nu = 2500$ Hz.

[1] Particles within the loss cone may well travel all the way to the earth's surface, causing the Aurora Borealis (Northern Lights) in polar regions.

# Chapter 3

# Plasmas as Fluids

## 3.1 Introduction

For 80% or so of the applications in plasma physics, it is sufficient to treat the plasma as a fluid of charged particles, in which electromagnetic forces are taken into account. The plasma fluid equations are a relatively simple modification of the Navier-Stokes equations; they must be supplemented by Maxwell's equations of electromagnetism, and the requirements of conservation of mass and charge.

Before discussing plasma as a fluid, I shall discuss the bulk properties of plasma (considered as a material medium) in electric or magnetic fields, and shall compare these bulk properties with the properties of ordinary matter. We begin by considering an external electric field, set up by placing a charge on external capacitor plates. When a slab of matter lies between the capacitor plates, the effect of the external electric field $\vec{E}_{ext}$ is to polarize the charges in the slab, so that opposite charges are induced on the faces of the slab nearest to to a respective capacitor plate, as shown.

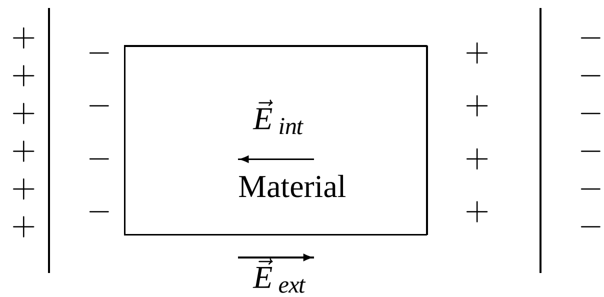


Figure 3.1: Material inside Capacitor

In other words, there is an induced electric field that is opposite to the external electric field. The net electric field inside the material is always sf less than the external field. One writes Gauss's law as

$$\varepsilon_0 \oint \vec{E} \cdot \vec{dS} = q_{enc} = q_{ext} + q_{ind} \tag{3.1}$$

The induced charge within a particular Gaussian surface, $q_{ind}$, may be computed as the negative flux of the polarization field $\vec{P}$.

$$q_{ind} = -\oint \vec{P} \cdot \vec{dS} \tag{3.2}$$

the polarization $\vec{P}$ being the induced electric dipole moment per unit volume. One may write Gauss's law in terms of the electric displacement

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

where $\vec{E} = \vec{E}_{net}$, as

$$\oint \vec{D} \cdot \vec{dS} = q_{ext} \tag{3.3}$$

The electric displacment $\vec{D}$ couples only to external charges.l For most materials (but certainly not for ferroelectric materials!), the polarization $\vec{P}$ is proportional to $\vec{E}$, so that

$$\vec{D} = \varepsilon \vec{E}$$

with the dielectric constant $\varepsilon$ dependent only on the state of the material. In particular, for a perfect conductor the electric fields are shielded out, so that $\vec{E} = 0$ and, in effect, $\varepsilon = \infty$.

A plasma is like a perfect conductor, in that it shields out static electric fields; the charges within the plasma rearrange themselves so thaat the $\vec{E}$-field is zero except within approximately one Debye length of the edges of the plasma. Debye shielding is effective only if there is no magnetic field in the plasma; in the presence of a $\vec{B}$ field the shielding is less complete, as we shall see.

If a magnetic induction $\vec{B}$ is present, the charged particles spiral around the field lines with frequency

$$\omega_c = qB/m$$

If, in addition, a transverse electric field

$$\vec{E}\ exp(i\omega t)$$

is present, the charged particles undergo a polarization drift in the direction of the change of $\vec{E}$.

If $\omega$ the frequency of variation of $\vec{E}$, is small in comparison with the cyclotron frequency $\omega_c$, we get a drift velocity

$$\vec{v}_D = \pm \frac{1}{\omega_c B} \frac{d\vec{E}}{dt} = \pm \frac{m}{qB^2} \frac{d\vec{E}}{dt} \tag{3.4}$$

Note: the drift speed is roughly $\pm i\omega E/\omega_c B$. With this drift speed in the direction of the transverse electric field, one may associate a polarization current (flow of electric charge) with the motion of ions and electrons

$$\vec{J} = ne\left(\vec{V}_{ions} - \vec{v}_{elec}\right) = \frac{n}{B^2}(m_e + m_i)\frac{d\vec{E}}{dt} = \frac{\rho}{B^2}\frac{d\vec{E}}{dt} \tag{3.5}$$

where

$$\rho = n(m_e + m_i)$$

is the mass density of the plasma. This current is associated with the drift of electric charges, and the locations of the electric charges must be taken into account in Gauss's law, Eq. (3.1). From the requirement of conservation of the flow of charge, applied to a Gaussian surface, it follows that

$$-\frac{d}{dt}q_{ind} = \oint \vec{J}\cdot\vec{dS} = \frac{\rho}{B^2}\frac{d}{dt}\oint \vec{E}_{ind}\cdot\vec{dS} \tag{3.6}$$

Thus, for low frequency time-dependent electric fields we have

$$q_{ind} = -\frac{\rho}{B^2}\oint \vec{E}\cdot\vec{dS} \tag{3.7}$$

On comparing (3.7) with (3.2) and the definition of the dielectric constant ε and the electric displacement, we obtain

$$\varepsilon = \varepsilon_0 + \frac{\rho}{b^2} \tag{3.8}$$

For a rather diffuse plasma with a stron electric field, $n = 10^{10}$ partiles per cubic centimeter and $B = 1$ kG, $\varepsilon/\varepsilon_0 = 200$, and in general ε increases with an increase in density and a decrease in magnetic field strength. Therefore, AC fields are fairly well shielded out by a plasma. The expression (3.8) is valid for transverse low-frequency only; in general the dielectric constant has rather complicated dependence on direction and frequency.

We shall now discuss the bulk properties of plasma, in contrast to those of ordinary matter. Ampère's law for ordinary matter has the form

$$\frac{1}{\mu_0} \oint \vec{B} \cdot \vec{d\ell} = i_{enc} = i_{ext} + i_{ind} \tag{3.9}$$

where the induced current is related to the magnetization $\vec{M}$ (magnetic dipole moment per unit volume) by the relation

$$i_{ind} = -\oint \vec{M} \cdot \vec{d\ell} \tag{3.10}$$

The magnetic field $\vec{H}$ is defined by the relation $\vec{B} = \mu_0 (\vec{H} + \vec{M})$, where the $H$-field couples only to external currents:

$$\oint \vec{H} \cdot \vec{d\ell} = i_{ext} \tag{3.11}$$

For a material medium we define the magnetic permeability $\mu$ by the relation

$$\vec{B} = \mu \vec{H}$$

A plasma must be considered as a diamagnetic medium, $\mu < \mu_0$, since the magnetic induction $\vec{B}_{in}$ generated by the spiraling of the charges is opposite in direction to the external induction $\vec{B}_{ext}$. The magnetic dipole moment of one spiraling particle is

$$\mu = m v_\perp^2 / (2B)$$

see Eq. (2.19). The magnetization is obtained by multiplying the average value of $\mu$ by $n$, the number of particles per unit volume:

$$M = \frac{m n v_{0\perp}^2}{2B} \tag{3.12}$$

for each species of charged particle in the plasma. Since the magnetization varies inversely as the magnetic induction, there is usually no advantage in treating the plasma as a magnetic medium. That is, one usuallly works directly with Ampère's law in the form (9), rather than the form (11) that couples only to external currents.

## 3.2 Fluid Equations

In the fluid approximation, one treats the positive ions, the negatively charged electons, and the neutral particles each as fluid components. The plasma is a mixed three-component fluid. The fluid equations include the effect of collisions and interactions of the fluid components, but do not take proper account of motions of the individual particles. In an ordinary fluid, the particles undergo frequent collisions, and these collisions justify the "fluid average" approximation. In plasma, the particles collide somewhat infrequently, but the magnetic field produces a similar averaging effect. That is, the magnetic field constrains the transverse motion of the charged particles, and thereby helps in the statistical averaging process. The fluid approximation is less valid in regard to longitudinal motions in plasma.

To illustrate the rather similar effects of collisions and the magnetic field, let us discuss the flow of curent in response to an electric field $\vec{E}$. In a copper wire, the electrons have drift velocities

$$\vec{v}_d = \mu \vec{E}$$

where the electron mobility is determined by the frequency of collisions in a metal. In plasma, there is a drift velocity

$$\vec{v}_d = \vec{E} \times \vec{B}/B^2$$

In the fluid approximation, one assumes that each infitimesimal clump of fluid moves without breaking up, so that one may associate a velocity with each clump of fluid. (We ignore the fact that the individual molecules in the clump have different velocities.) The fluid equations of motion for the plasma, which are analogous to the Navier-Stokes fluid equations, are obtained by applying Newton's second law to an infinitesimal clump of fluid of mass $m$ and charge $q$:

$$m \frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + \text{collisional forces} \tag{3.13}$$

The term $d\vec{V}/dt$, which is the acceleration of a particular clump of fluid, is not simply the rate at which the velocity is changing at a particular point in space. This term is somewhat inconvenient both to measure and to deal with analytically, since one must follow each fluid clump to determine it. Instead, it is conventional to make use of the velocity field $\vec{u}(\vec{r},t)$, which gives the velocity of fluid at position $\vec{r}$ and time $t$. The fluid clumps experience accelerations when the velocity field $\vec{u}$ is nonuniform in space or in time; in fact the acceleration $d\vec{v}/dt$ is the so-called convective time derivative of the velocity field, $D\vec{u}/Dt$:

$$\begin{aligned}\frac{D\vec{u}}{Dt} &= \frac{\partial\vec{u}}{\partial t}+\frac{\partial\vec{u}}{\partial x}\frac{dx}{dt}+\frac{\partial\vec{u}}{\partial y}\frac{dy}{dt}+\frac{\partial\vec{u}}{\partial z}\frac{dz}{dt}\\ &= \frac{\partial\vec{u}}{\partial t}+\frac{\partial\vec{u}}{\partial x}v_x+\frac{\partial\vec{u}}{\partial y}v_y+\frac{\partial\vec{u}}{\partial z}v_z \end{aligned} \tag{3.14}$$

Remark: The convective time derivative not just the ordinary time derivative of a field. Your house may be losing heat (nonzero convective derivative of temperature with respect to time) even though the thermometer reading on the thermostat does not change.

Let us apply Eq. (13) to an infinitesimal volume $dV$, containing $n\,dV$ charged particles, and divide it by $dV$. We obtain

$$mn\frac{D\vec{u}}{Dt} = qn\left(\vec{E}+\vec{v}\times\vec{B}\right)+\text{pressure force per unit volume} \tag{3.15}$$

The pressure force in a fluid can be computed for a cubic fluid element, with pressure $p_1$ (down) on the top face and pressure $p_2$ (up) on the bottom face. The bouyant force (up) is

$$(p_2-p_1)A=(p_2-p_1)V/h$$

Consequently, the force per unit volume on a small cube is equal to $-dp/dx$, the negative of the pressure gradient.

The fluid equaiton (15) must be supplemented by the equation of state of the plasma – one must know how the pressure of plasma changes with its density. A dilute plasma in thermal equilibrium, like a dilute gas of neutral particles or a dilute solute, obeys the perfect gas law

$$pV = Nk_BT \quad \text{or} \quad p = nk_BT$$

As with the gas or the solute, there are two very common ways in which the pressure changes with density:

1. Isothermal changes – the chnages in density are sufficiently gradual that the system remains in thermal equilibrium at temperature $T$, so $p$ is proportional to $\rho$.

2. Adiabatic changes – changes in the state of the plasma often occur over times too short for thermal equilibrium to be established; such changes can be treated as adiabatic (sudden).

For adiabatic changes in the state of an ideal gas, $pV^\gamma$ is an invariant. The plasma may usually be treated for such purposes as a one-dimensional gas of point particles, so that $\gamma = 3$. the appropriate pressure-density relation is

$$p = c\rho^\gamma \tag{3.16}$$

The pressure increases with density for adiabatic, as well as for isothermal, processes.

Remark: The plasma will often be in thermal equilibrium separately for longitudinal and transverse motions, with different temperatures. That is, the plasma my have time to establish separate thermal equilibrium with respect to longitudinal and transverse motions, but there may not be enough time for these independent degrees of freedom to come into mutual equilibrium. The Maxwell-Boltzmann distribution of speeds would be non-isotropic in such cases, with different "temperatures" for transverse and longitudinal directions.

Let us now discuss the fluid motions of plasma in external electric and magnetic field. We begin by talking about transverse motions, perpendicular to $\vec{B}$. The fluid equation is

$$mn\frac{D\vec{u}}{Dt} = qn\left(\vec{E} + \vec{u}\times\vec{B} + \frac{1}{qn}\text{grad}\, p\right) \tag{3.17}$$

The structure of this equation for transverse motions is very similar to those single-particle equations discussed in Chapter 2. The term $\vec{u}\times\vec{B}$ causes Larmor gyration of the fluid, whereas the other two terms induce fluid drifts. The second term gives rise to an "$\vec{E}\times\vec{B}$ drift" in the fluid, with drift speed

$$\vec{v}_D = \frac{1}{B^2}\vec{E}\times\vec{B} \tag{3.18}$$

and the third term produces a "pressure gradient drift", or "diamagnetic drift" with speed

$$\vec{v}_D - \frac{1}{qmB^2}\,\text{grad}\, p\times\vec{B} \tag{3.19}$$

This $\vec{E} \times \vec{B}$ drift is identical with that encountered in the single particle case. To understand the physical basis for diamagnetic drifts, consider a plasma in which the pressure gradient goes to the left and the $\vec{B}$ field points out of the paper. The plasma is more tenuous to the right, and fewer particles are moving in cyclotron orbits. In the strip shown, more particles enter from the left, and move downward, than enter from the right to move upward. Therefore, there is a new downward drift of fluid in the strip. (See Figure 3.2.)

In practical units, for isothermal drifts of electrons,

$$\vec{v}_D = 10^8 \frac{k_B T}{L B} \text{ cm/sec} \tag{3.20}$$

where $T$ is the transverse electron temperature, with $k_B T$ in electron Volts, $B$ is the magnetic induction in Gauss, and $L$ is the length scale (in cm) over which density changes occur in the plasma.

Gradient $B$ drifts and curvature drifts do not occur in the fluid picture. These drifts are absent here, since the magnetic induction does not work on charged particles, and the presence or absence of a $\vec{B}$ field does not affect the Maxwellian distribution of speeds. In other words, although the guiding centers of the individual particles do experience drifts, these drifts are presumably cancelled out by collective fluid effects. The fluid picture and the single particle picture are somewhat contradictory on the matter of these drifts.

For fluid drifts in the direction parallel to the $\vec{B}$ field (the $z$-direction) the fluid equation for the more mobile electons becomes

$$m n \frac{D\vec{u}_z}{Dt} = q n E_z - \frac{dp}{dz} \tag{3.21}$$

Because the electrons are light, they very quickly rearrange themselves so that the pressure gradient term in (3.21) cancels out the external electric field. Furthermore, such rearrangements occur so quickly that thermal equilibrium is established. Thus,

$$q n E_z = \frac{dp}{dz} = \int k_B T \frac{dn}{dz} \tag{3.22}$$

We can relate the left side of Eq. (3.22) to the electric potential $\phi$ by writing it as

$$(-e)\, n\, (-d\phi/dz) = e\, n\, (d\phi)(dz)$$

The number density of the electrons is therefore

$$n(z) = n_0 \exp[\frac{e\phi}{k_B T}] \tag{3.23}$$

The electrons are attracted to the regions of high electric potential, and pushed awan from the regions of low potential; the pressure gradients of the electrons cancel out the external electric fields.

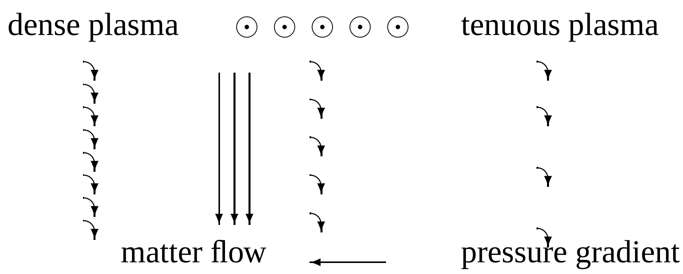


Figure 3.2: Diamagnetic Drift

# Chapter 4

# Waves in Plasmas

## 4.1 Introduction

We shall be concerned with certain disturbances from equilibrium in a plasma propagate in space as time passes. These disturbances are wavelike, and similar in spirit to sound waves, water waves, earthquake waves, and electromagnetic waves. Before discussing various distinct types of waves in plasma, we briefly review wave phenomena in general. For simplicity, we shall discuss waves in one-dimensional space – the generalization to three-dimensional space is direct and obvious to those readers with the appropriate background.

Let $f(x,t)$ be the "wave disturbance" at position $x$ and time $t$. ($f$ might be the displacement of particles from equilibrium, or the magnitude of the electric field, in a physical situation.) For a plane wave $f$ is a sinusoidally varying function throughout space and time:

$$f(x,t) = Re\left(f_0\, e^{i\phi}\, e^{i(kx-\omega t)}\right) = f_0 \cos(kx - \omega t + \phi) \tag{4.1}$$

where the following real variables appear in (4.1):

- $f_0$: wave amplitude
- $\phi$: phase of wave (at $x = 0 = t$)
- $k$: wave number
- $\omega$: angular frequency

Plane waves are unphysical, in that they persist throughout all space and time; we discuss them merely because they are the simplest solutions of the wave equations. Real waves, or "wave packets" are localized in space; we may represent them as superpositions of plane waves.

We introduce the concept of superposition by taking the sum of two wave disturbances of equal amplitude $f_0$ and almost the same wave number and frequency: [$(\omega_1, k_1)$ and $(\omega_2, k_2)$, respectively.] We obtain the net disturbance

$$f_0 \cos(k_1 x - \omega_1 t + \phi_1) + f_0 \cos(k_2 x - \omega_2 t + \phi_2) = \tag{4.2}$$
$$2 f_0 \cos\left[\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} + \frac{\phi_1 + \phi_2}{2}\right] \cos\left[\frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} + \frac{\phi_1 - \phi_2}{2}\right]$$

The appearance of two separate cosines on the right side is characteristic of superposition. The first factor represents the carrier wave. That factor remains fixed whenever

$$\bar{k} x - \bar{\omega} t$$

does not change, where $\bar{k}$ is the average wave number and $\bar{\omega}$ is the average angular frequency. In other words, the carrier wave propagates with velocity equal to the phase velocity

$$v_P = \bar{\omega} / \bar{k} \tag{4.3}$$

The second cosine factor represents the propagation of energy and information in the wave; it remains fixed whenever

$$\Delta k x - \Delta \omega t$$

does not change, with $\Delta k$ the spread in wave number and $\Delta\omega$ the angular frequency spread. Consequently, the energy of the wave propagates with the group velocity

$$v_g = \frac{\Delta \omega}{\Delta k} \rightarrow \frac{d\omega}{dk} \tag{4.4}$$

We shall use Maxwell's equations and the fluid equations to identify various types of disturbances propagating in plasma, and to obtain the dispersion formula to give angular frequency $\omega$ as a function of the wave number $k$ for these various types of plasma waves.

As discussed in Chapter I, a charge imbalance in a plasma sets up an electric field, and the charges subsequently oscillate about their equilibrium positions with the plasma frequency

$$\omega_P = \sqrt{\frac{n e^2}{\varepsilon_0}} \approx 9000 \sqrt{n} \text{ rad/sec} \tag{4.5}$$

where $n$ is the number of particles per cubic centimeter in the last term. The plasma oscillation is the key to propagation of waves in plasma, in the same sense that molecular motion is the key to propagation of sound in matter. However, the oscillations do not set up a propagating distrubance in an infinite plasma, since $\omega_P$ is independent of $k$, and the corresponding group velocity $v_g = d\omega/dk$ is zero.[1]

## 4.2 Electron Plasma Waves

We shall first discuss electron plasma waves, which are electrostatic waves that propagate in a plasma with no external magnetic field. A disturbance propages because of the thermal motion of the electrons; one gets a condensation-rarefaction electron density wave in the plasma. [2]

To discuss electron plasma waves in detail, we write the electron number density $n(x,t)$ as

$$n(x,t) = n_0 + n_1(x,t) \tag{4.6}$$

where $n_0$ is the uniform equilibrium electron density[3] and $n_1$ represents the propagating wave density. The fluid equation for the electrons relates the convective derivative of the velocity field $\vec{v}$ to the electric field $\vec{E}$ and the pressure gradient:

$$m\,n\,\frac{D\vec{v}}{Dt} = -e\,n\,\vec{E} - \mathrm{grad}\,p \tag{4.7}$$

We assume that $n_1$, the deviation from uniform electron density, and $\vec{v}$, the velocity field of the electrons, are small, and keep only first order terms to obtain

$$m\,n_0\,\frac{DV_x}{Dt} = -e\,n_0 E_x - \frac{dp}{dx} \tag{4.8}$$

If we assume the wave propagates adiabatically, the pressure gradient may be replaced by $3k_B T\,dn_1/dx$. Let us take $n_1$ and $V_x$ to have the space-time dependence of a plane wave:

$$n_1(x,t) = n_1 e^{i(kx-\omega t)} \qquad\qquad v_x(x,t) - v_x e^{i(kx-\omega t)} \tag{4.9}$$

We put (4.9) into (4.8) to obtain

$$i m n_0\,\omega\,v_x = e\,n_0 E_x + 3i\,k_B T\,n_1 \tag{4.10}$$

[1]In an infinite plasma, there is a propagating disturbance because of fringing fields – but never mind that.

[2]If the electrons had no thermal motion, the usual localized plasma oscillation would occur.

[3]We are pretending in this chapter that particles are of uniform density in the plasma!

The fluid equation (4.8) must be supplemented by Gauss's law and the equation of continuity, or law of conservation of the flow of electrons, in determining the dispersion relation $\omega(k)$ for these electron plasma waves. We discuss both of them in terms of a closed transverse cylindrical surface of cross-sectional area $A$, with faces at longitudinal faces at 0 and $x$, as shown in Figure 4.1.

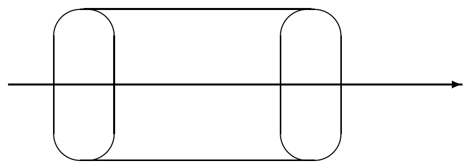


Figure 4.1: Cylindrical Gaussian Surface

By symmetry, the electric field has an $x$-component independent of $y$ and $z$, so that the flux of the electic field is

$$(E_x(x) - E_x(0))A$$

The net charge density inside the cylinder is $-en_1$. (We have assumed that the plasma in bulk is electrically neutral, with the ion charge density to cancel out the background electric charge density $-en_0$. Gauss's law yields the relation

$$\varepsilon_0 (E_x(x) - E_x(0))A = -eA \int_{)}^{x} dx\, n_1$$

or

$$\varepsilon_0 \frac{dE_x}{dx} = -n_i e \tag{4.11}$$

The law of conservation of mass requires that the negative time derivative of the amount of mass inside equals the flux of mass through the surface. There is no flux through the lateral surface, so that

$$-\frac{d}{dt} \int_0^x (nmA) dx = (n(x)\, v(x) - n(0)\, v(0))\, mA \tag{4.12}$$

We divide by $mA$ and differentiate with respect to $x$ to obtain

$$-\frac{dn}{dx} = \frac{d}{dx} (nv_x) \tag{4.13}$$

or, to first order,

$$\frac{dn_1}{dx} = -n_0 \frac{dv_x}{dx} \tag{4.14}$$

We use the plane wave expression (4.9) in (4.11) and (4.14), as well as a similar expression for the electric field $E_x$, to obtain

$$\begin{aligned} ik\varepsilon_0 E_x &= -n_1 e \\ i\omega n_1 &= ikn_0 v_x \end{aligned} \tag{4.15}$$

We then use (4.15) to eliminate $E_x$ and $n_1$ in the fluid equation (4.10).

$$im\omega v_x = en_0 \frac{-e}{ik\varepsilon_0} \frac{kn_0 v_x}{\omega} + 3k_B T(ik) \frac{kn_0 v_x}{\omega} \tag{4.16}$$

or

$$\omega^2 = \frac{n_0 e^2}{m\varepsilon_0} + \frac{3k_B T}{m} k^2 = \omega_P^2 + \frac{3k_B T}{m} k^2 \tag{4.17}$$

The dispersion curve of $\omega$ versus $k$ is one branch of a hyperbola.

For the group velocity $v_g = d\omega/dk$ we have

$$v_g = \frac{3k_B T}{m} \frac{k}{\omega} \leq \sqrt{\frac{3k_B T}{m}} = v_{th} \tag{4.18}$$

It is always less than the thermal speed, and approaches it at large wave number. At small wave number, the pressure gradient term in (4.8) is small, and the group velocity is small in comparison with the thermal speed. By contrast, the phase velocity is always larger than the thermal speed, and becomes infinite as $k$ approaches zero.

The experiments of Looney and Brown (1954) established the existence of electron plasma waves.

The discussion of electron plasma waves is similar to that of sound waves in an ideal gas, except that the electric field plays a role. We recover the expression for transmission of sound waves by making these changes in the above expression:

- Since the atoms of the gas are neutral, we set the charge $e$ and the electric field $E_x$ equal to zero.

- The factor of $3k_B T$ in (4.10), (4.16), and (4.17) should be replaced by $\gamma k_B T$, where $\gamma$ is the ratio of specific heats for the gas in question.

The sound wave propagates because of atomic collisions, and the dispersion relation is

$$\frac{\omega}{k} = \sqrt{\frac{3k_B T}{m}} \tag{4.19}$$

(Of course, $m$ is the mass of the atom itself.)

## 4.3 Ion Acoustic Waves

Next we discuss Ion Acoustic Waves, which are the plasma physics version of sound waves. These waves propagate because the more mobile electrons shield out the ion electric field, but set up their own field to drive the ions in a tenuous plasma. These waves can be excited, even though the electrons and ions have infrequent collisions. The external magnetic field is zero for these waves.

The ion fluid equation has the form

$$Mn\frac{D\vec{v}_i}{Dt} = en,\vec{E} - \mathrm{grad}\,p = -en\,\mathrm{grad}\,\phi - \gamma_i k_B T_i\,\mathrm{grad}\,n \tag{4.20}$$

We have introduced the electrostatic potential $\phi$ and assumed adiabatic changes in the ion state in obtaining this relation. We make the plasma approximation in discussing ion waves; that is, we assume that the more mobile electrons shield out the electric field produced by the ions; so that the electron and ion densities are equal to one another and have the value

$$n_i = n_e = n_o \exp\left[\frac{e\phi}{k_B T_e}\right] \tag{4.21}$$

where $\phi$ is the electric potential at the point in question. If we assume the electrostatic energy to be small compared to $k_B T_e$, the ion density may be written as

$$n_i = n_0\left(1 + \frac{e\phi}{k_B T_e}\right) \tag{4.22}$$

The uniform background density of electrons and ions is $n_0$.

Let us substitute plane wave forms for $n_i$, $\vec{v}$, and $\phi$ into the fluid equation (4.20) to obtain the first order equation

$$i\omega n_o v_{ix} = en_0(ik\phi) - \gamma_i k_B T_i(ikn_1) \tag{4.23}$$

Finally, we need the ion equation of continuity (4.13), which gives

$$i\omega n_1 = ikn_0 v_{ix} \tag{4.24}$$

We eliminate $n_1$ and $\phi$ to obtain

$$i\omega M n_0 v_{ix} = \frac{ikn_0 v_{ix}}{\omega} ik(k_B T_e) + \frac{kn_0 v_{ix}}{\omega} ik(\gamma_i k_B T_i) \tag{4.25}$$

or

$$\frac{\omega^2}{k^2} = \frac{k_B T_e}{M} + \gamma_i \frac{k_B T_i}{M} \tag{4.26}$$

In a practical context, the electron temperature $T_e$ is often much larger than the ion temperature $T_i$, so that the first term on the right dominates (4.24). In other words, the dominant mechanicsm for wave propagation is the residual electric field produced by the thermal motion of electrons about equilibrium. The ion acoustic wave propagates because of the thermal agitation of the electrons (an isothermal process – the electrons have time to come to thermal equilibrium before the ions move) and the thermal agitation of ions (an adiabatic process).

Let us now turn to the discussion of electrostatic oscillations of a plasma in a uniform magnetic field $\vec{B}$ – in particular, we discuss the oscillations of electrons in the plane perpendicular to $\vec{B}$. Such oscillations are produced by two distinct physical mechanisms:

- plasma oscillations of frequency $\omega_P$
- Larmor gyration of electrons in the magnetic field, of frequency

$$\omega_c = eB/m$$

The "upper hybrid frequency" for such oscillations is $\omega$, where

$$\omega^2 = \omega_P^2 + \omega_c^2 = \left(\frac{ne^2}{m\varepsilon_0}\right)^2 + \left(\frac{eB}{m}\right)^2 \tag{4.27}$$

The plasma frequency and the cyclotron frequency are of comparable magnitudes in a typical laboratory plasma. The frequency is greater than either $\omega_P$ or $\omega_c$, since the electron restoring force and the Lorentz force are added to give the total restoring force, which is greater than each force separately.

The electromagnetic oscillations do not themselves propagate, because $\omega$ is independent of the wave number $k$, but they and other plasma oscillations do provide a mechanism for propagation of waves in plasma. We mention in passing one example of such a propagating wave: electrostatic ion acoustic waves, which propagate in the direction perpendicular to the external field $\vec{B}_0$, and are subject to the dispersion relation[4]

$$\omega^2 = \left(\frac{eB_0}{m}\right)^2 + k^2\,\frac{k_B T_e}{m} \tag{4.28}$$

The ion cyclotron frequency plays the same role here as the electron plasma frequency in electron plasma waves; see Eq. (4.17).

## 4.4 Electromagnetic Waves in Plasma

Let us now turn the discussion to the propagation of electromagnetic waves in plasma. We begin with electromagnetic electron waves, with no external fields. The electrons in the plasma are accelerated by the electromagnetic field of the incident wave, and the electrons, in turn, produce $\vec{E}$ and $\vec{B}$ fields on their own that modify the form of the wave. The dispersion relation for these plasma waves is

$$\omega^2 = \left(\frac{n\,e^2}{m\varepsilon_0}\right)^2 + c^2\,k^2 = \omega_P^2 + c^2\,k^2 \tag{4.29}$$

Note that the group velocity of such waves, $v_g = c^2\,k/\omega$, is always less than the velocity of light $c$, so that the presence of plasma in a region serves to reduce the speed of propagation of electromagnetic radiation. Note also that, since $\omega$ is not merely a linear function of $k$, there is dispersion of a wave packet in plasma – components with different wave numbers propagate with different speeds. Finally, note that if $\omega$ is less than the plasma frequency $\omega_P$, the wave number $k$ must be purely imaginary – such waves are exponentially damped in the plasma and cannot propagate through it. As electromagnetic wave that comes to a region in space in which $\omega < \omega_P$ is, in general, reflected at the boundary. The physics is the same as that of total internal reflection, the process by which light is transmitted through lucite fibers, for example.

The earth's ionosphere has a density of about $10^6$ charged particles per cubic centimeter, and the plasma frequency is about $10^7$ Hz. Electromagnetic waves

[4](We have neglected the ion temperature here.

of below that frequency (e. g., AM radio waves) are reflected by the ionosphere, whereas waves of higher frequency (FM and TV, for example) are transmitted through the ionosphere. One may communicate over great distances with short waves by making reflections off the earth's ionosphere. On the other hand, communications with satellites in the ionosphere and beyond[5] must employ electromagnetic waves of higher frequency. The plasma cutoff, or blackout of communications of satellites upon re-entry through the earth's atmosphere, is another manifestation of this attenuation of low-frequency electromagnetic waves in plasma.

We shall briefly mention electron electromagnetic waves in plasma with a uniform magnetic field $\vec{B}_0$. It is possible for waves to propagate in a direction perpendicular to $\vec{B}_0$ – there are direction-dependent effects, just as for electromagnetic waves propagating in an anisotropic crystal. In addition, electromagnetic waves also propagate in a direction parallel to $\vec{B}_0$. Because the electrons and ions in a plasma undergo Larmor gyration in the external magnetic field, there is a natural asymmetry, so that left and right circularly polarized waves have different dispersion relations. In particular, if we take the electrical conductivity of the plasma into account, waves with one sense of polarization are attenuated over shorter distances than those with the other sense. With $\vec{B}_0 = 0$, waves are in general attenuated rapidly because of the effect of conductivity. These readily transmitted waves are called "helicons" in solid state physics and "whistlers" in plasma physics.

When lightning flashes occur in the Southern hemisphere, a broad spectrum of electromagnetic radiation propagates along magnetic field lines (without substantial attenuation), coming back to earth in the Northern latitudes where the field lines return to the earth. The group velocity of these waves increases with frequency, so that the high frequency waves arrive first and th low frequency waves arrive later. (These waves are readily converted into an audio signal by "superheterodyning".) The characteristic "chirp" or whistle of these signals (i. e., a shift from high pitch to low pitch) gives these signals their name. Whistlers were first detected by primitive military shortwave communications in Normandy in World War I.

[5] and with little green men in outer space via ESP

## 4.5 Alfvén Waves

Finally, we discuss low frequency ion electromagnetic waves in the presence of an external magnetic field; in particular, we consider hydromagnetic waves, or Alfvén waves, which propagate parallel to $\vec{B}_0$. The electron and ion motions lie in the plane perpendicular to the wave number $\vec{k}$. The dispersion relation for electromagnetic waves in a material medium with dielectric constant $\varepsilon$ is

$$\frac{\omega^2}{k^2} = \frac{c^2}{\varepsilon/\varepsilon_0} \tag{4.30}$$

We may adapt this formula to plasmas, taking $\varepsilon$ to be the transverse dielectric constant in an external magnetic field $\vec{B}_0$ for a plasma of density $\rho$. See Eq. (3.8).

$$\varepsilon = \varepsilon_0 + \frac{\rho}{B_0^2} \tag{4.31}$$

In general, $\varepsilon \gg \varepsilon_0$, and one has approximately

$$\frac{\omega^2}{k^2} = \frac{c^2\,\varepsilon_0\,B_0^2}{\rho} \equiv v_A^2 \tag{4.32}$$

where $v_A$ is the Alfvén velocity.

The physical explanation of Alfvén waves is that a current $j_1$ is produced in the direction of the induced electric field $\vec{E}_1$ of the electromagnetic waves, as a result of Ohm's Law. The elctrons and the ions experience a drift $\vec{v}_1 = \vec{E}_1 \times \vec{B}/B^2$ – this is just the $\vec{E} \times \vec{B}$ drift. The current $j_1$ experiences a Lorentz force

$$\vec{F} = \vec{j}_1 \times \vec{B}$$

which serves to cancel the $\vec{E} \times \vec{B}$ drift, and produce oscillations. The transverse electric and magnetic fields are related by Faraday's Law:

$$E_1 = \omega B_1\, k$$

We mention that, among many other varieties of ion electromagnetic waves, there are magnetosonic waves, which are acoustic waves propagating perpendicular to $\vec{B}_0$. Compressions and rarefactions of plasma are produced by $\vec{E} \times \vec{B}$ drifts along the wave vector $\vec{k}$.

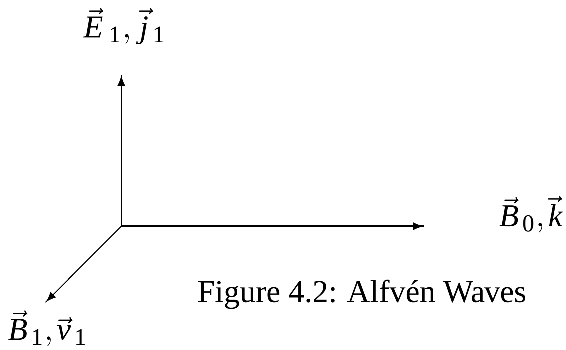


Figure 4.2: Alfvén Waves

# Chapter 5

# Diffusion, Equilibrium, Stability

## 5.1 Introduction

We have been discussing plasma of uniform density up to now, but in actual laboratory practic plasmas are certainly not of uniform density. There is certain to be a net movement (diffusion) of particles from high-density regions to low-density regions (e. g., to the walls). In laboratory confinement of plasma for the purpose of studying or inducing controlled thermonuclear reactions, the magnetic field is used to impede the process of diffusion in a weakly ionized plasma, in which the relevant equations are linear, and therefore simpler than in the strongly-ionized case.

The fluid equation for the charged particle velocity field is

$$mn\frac{D\vec{v}}{Dt} = \pm en, \vec{E} - \operatorname{grad} p - mn\nu\vec{v} \tag{5.1}$$

The first two force terms come from electric fields and pressure gradients, whereas the third is a collisional term. The parameter $\nu$ is the collision frequency multiplied by the fraction of momentum lost by a charged particle in an average collision. Let us apply this equation to a steady-state diffusional process, so that $\partial\vec{v}/dt = 0$. Under the additional assumption that the diffusional velocity $\vec{v}$ is small, we may set the convective derivative $D\vec{v}/Dt$ equal to 0. Finally, we assume the diffusion to take place under isothermal conditions: $\operatorname{grad} p = -k_B T \operatorname{grad} n$. One may then solve Eq. (5.1) for the velocity field:

$$\vec{v} = \frac{1}{mn\nu}\left(\pm en\vec{E} - k_B T \operatorname{grad} n\right) = \pm\frac{e}{m\nu}\vec{E} - \frac{k_B T}{mn}\frac{\operatorname{grad} n}{n} \tag{5.2}$$

The coefficient of $\vec{E}$ is the mobility $\mu$ of the charged particle, and the coeffient of grad $n$ in the second term is the coefficient of diffusion $D$.

The particle flux at a point in space, which is defined as the number of particles passing through a unit area per unit time, is $\vec{\Gamma} = n\vec{v}$. We thus have

$$\vec{\Gamma} = \pm\mu n\vec{E} - D\,\mathrm{grad}\,n \tag{5.3}$$

A particle flux can be independently produced as a result of an electric field or a density gradient. We have made the tacit assumption that diffusion is produced as a result of random motions of the particles. However, since the motions in an external magnetic field are not random, the above considerations may require modification in certain cases.

## 5.2 Ambipolar Diffusion

We shall next discuss the diffusion of a quasi-neutral plasma of electrons and ions, which is called ambipolar diffusion. The electrons are more mobile than the more massive ions – they have greater diffusion constants and tend to diffuse first to less dense regions. The initial movement of electrons, however, creates a charge imbalance, and thus an electric field, which retards the diffusion and enhances the diffusion of ions. Under steady-state conditions electrons and ions diffuse at the same rate. They have the same flux $\Gamma$ and the same density $n$, and there is no charge imbalance. According to Eq. (5.3),

$$\vec{\Gamma} = \mu_i n\vec{E} - D_i\,\mathrm{grad}\,n = -\mu_e n\vec{E} - D_e\,\mathrm{grad}\,n \tag{5.4}$$

so that the electric field that retards electrons is

$$\vec{E} = -\frac{D_e - D_i}{\mu_i - \mu_e}\,\frac{\mathrm{grad}\,n}{n} \tag{5.5}$$

and the flux is

$$\vec{\Gamma} = -\frac{D_e\mu_i + D_i\mu_e}{\mu_i + \mu_e}\,\mathrm{grad}\,n \tag{5.6}$$

As we see from (5.6), the rate of diffusion depends upon some “effective diffusion coefficient”. The effect of the induced electric field is to incrase the diffusion of ions. In general the slower species (ions) controls the overall diffusion rate.

We next discuss the rate at which plasma that is created within a vessel decays by diffusion to the walls. To discuss this problem we use the equation of continuity, which expresses the conservation of the flow of mass:

$$\frac{\partial}{\partial t}\int n\,dV + \oint(\vec{\Gamma}\cdot\hat{n})dS = 0 \tag{5.7}$$

When $n$ and $\vec{\Gamma}$ are dependent only on the $x$-coordinate, with only an $x$-component for $\vec{\Gamma}$, Eq. (5.7) is equivalent to

$$\frac{\partial n}{\partial t} = -\frac{\partial \Gamma_x}{\partial x} \tag{5.8}$$

We shall also need Fick's law of diffusion:

$$\vec{\Gamma} = -D\,\mathrm{grad}\,n$$

or

$$\Gamma_x = -D\partial n/\partial x$$

(It is a special case of Eq. (5.3).) Thus

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial n}{\partial x}\right) = D\frac{\partial^2 n}{\partial x^2} \tag{5.9}$$

The particles in the plasma diffuse to the walls, where they collide with the walls and lose their charge – the concentration $n$ of charged particles at the walls of the container is thus zero.

We determine $n(x,t)$ from the initial concentration $n(x,0)$ by solving Eq. (5.9) subject to the boundary condition $n = 0$ at the walls. We may express the solution as a linear combination of the diffusional modes, in the same sense that an arbitrary motion of an ideal string may be expressed as a superposition of normal modes. For the diffusional modes, we assume

$$n(x,t) = X(x)\,T(t)$$

and insert this expression into Eq. (5.9) to obtain

$$T(t) = \exp[-t/\tau]$$

with $X(x)$ satisfying the ordinary differential equation

$$\frac{d^2X}{dx^2} + \frac{1}{D\tau}X = 0 \tag{5.10}$$

The allowed values of the parameter $\tau$ are determined by finding all solutions of Eq. (5.10) that vanish at the walls – there are an infinite number of such allowed values and corresponding diffusional modes. In general, the characteristic diffusional time $\tau$ is of order $L^2/D$, where $L$ is a characteristic distance for changes of concentration, which is typically no larger than the size of the container.

One may describe a steady-state diffusional process, in which plasma is continually being ionized by a beam of high-speed electrons, by adding a source term proportional to the right side of (5.9), to obtain

$$\frac{d^2X}{dx^2} + ZX = 0 \tag{5.11}$$

This equation is identical in form to (5.10), although it describes a physically different case.

If the concentration of ions and electrons in the plasma becomes too great, the process of recombination becomes important. That is, the ions and electrons recombine within the plasma to form neutral atoms. when one neglects diffusion altogether, the decay of the plasma is determined by the relation

$$\frac{\partial n}{\partial t} = -\alpha n^2 \tag{5.12}$$

where $\alpha$ is determined from the rate constant for the deionization reaction $e^- +$ ion $\rightarrow$ neutral atom. The solution of (5.12) is

$$n(x,t) = \frac{n(x,0)}{1 + \alpha t\, n(x,0)} \tag{5.13}$$

In other words, the decay of the plasma at high density through recombination is proportional to $1/t$. At low density, the dominant mechanism is diffusion to the walls, corresponding to exponential decay with some characteristic diffusional time $\tau$.

We shall next discuss diffusion in a weakly ionized plasma in directions perpendicular to an external magnetic field $\vec{B}$. The presence of the magnetic field cuts down on diffusion, since particles spiral around field lines, instead of drifting in straight paths, between collisions. Naturally, one must deal with drifts caused by electric fields or non-uniform magnetic fields before diffusional drifts can be considered important. One can show, by a direct but lengthy argument that we shall bypass, that the effective transverse diffusional constant is

$$D_\perp = \frac{D}{1 + \omega_c^2 \tau^2} \tag{5.14}$$

under ideal conditions. The quantity $\tau$ is the time between collisions, and $\omega_c = eB/m$ is the cyclotron frequency. Typically, one has $\omega_c\tau \gg 1$, so that a particle makes many Larmor orbits between collisions. Under such conditions the magnetic field retards diffusion. In the case of ordinary diffusion, the diffusion constant $D$ is of order $\lambda_m^2/\tau$, where $\lambda_m$ is the mean distance between collisions. With a magnetic field present, $D_\perp$ is of order $r_\perp^2/\tau$, where $r_\perp$ is a typical Larmor radius.

The experiments of the Swedish physicists Lehnert and Hon verified that magnetic fields reduce diffusion. They actually obtained anomalously large diffusion at large $B$, since an electric field $\vec{E}$ parallel to $\vec{B}$ was induced, and there was a transverse $\vec{E} \times \vec{B}$ drift, htat resulted in enhanced drift. As we shall see, it is notoriously difficult to get rid of anomalous sources of diffusion.

To discuss the behavior of a fully ionized plasma in an external $\vec{B}$ field, one must take account of the Coulombic collisions of the particles, which are spiraling in the presence of $\vec{B}$, taken perpendicular to the paper in the diagram:

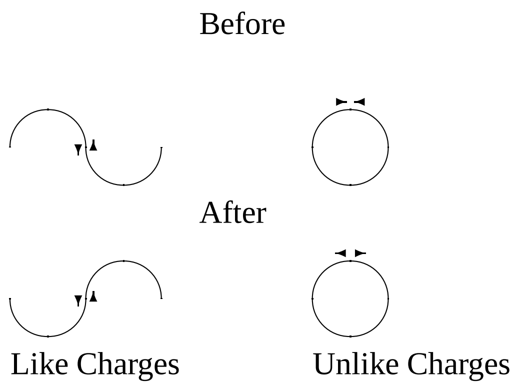


Figure 5.1: Drift upon Collisions

We see that for the collisions of like charges, there is no net shift in the guiding center positions, and thus no diffusion. For unlike charges the guiding centers have shifted, and there is diffusion. The electron-ion collisions[1] give rise to electrical resistivity in a plasma. The motion of charges gives rise to a current

$$\vec{j} = ne\vec{v}$$

If this drift is produced by an electric field, we use the first part of Eq. (5.2) to obtain

$$\vec{v} = \pm e\vec{E}/(m\nu)$$

[1] The collisional frequency of a particular electron with an ion is $\nu e - i$.

The resistivity is defined by the relation

$$\vec{E} = \eta \vec{j}$$

so that

$$\eta = \frac{m \nu_{e-i}}{n e^2} \tag{5.15}$$

From a detailed calculation, one actually obtains

$$\eta = \frac{5.2 \times 10^{-3}\,\text{Ohm cm}}{[T\,(\text{ev})]^{3/2}} \times \log \Lambda \tag{5.16}$$

where $\Lambda = \lambda_D / r_0$, $\lambda_D$ being the Debye length in the plasma; and $r_0$ is of order $e^2/(4\pi\varepsilon_0 k_B T)$, being the "maximum impact parameter" of the charges in the plasma. Note that the parameter $\lambda$ depends on the density, so that $\eta$ is weakly density-dependent. The resistivity is almost independent of density, since the collision frequency $\nu$ in Eq. (5.15) is proportional to $\eta$.[2]

If we ignore the weak temperature-dependence in $\log\Lambda$, then $\eta$ behaves as $T^{-3/2}$. In other words, the Coulomb cross-section decreases at high kinetic energies, or at high temperatures. High temperature plasmas are almost collisionless – one cannot heat them simply by passing a current through the plasma. Furthermore, the high speed electrons in such a plasma make very few collisions, so they carry most of the current. One may encounter the phenomenon of electron runaway, in which the high speed electrons continue to accelerate in the plasma, so that they can pass across the plasma without making a single collision.

The form of Ohm's law in the presence of a magnetic field is

$$\left(\vec{E} + \vec{j} \times \vec{B}\right) = \eta \vec{j} \tag{5.17}$$

One may use this relation and go through a lengthy argument to show that, in a stronglly ionized plasma, the transverse diffusion coefficient is

$$D_\perp = \frac{\eta n}{B^2}\,(k_B T_i + k_B T_e) \tag{5.18}$$

Note that $D_\perp$ is proportional to $B^{-2}$, as before. However, $D_\perp$ is proportional to the concentration $n$, rather than being independent of it. As a result, the decay

[2] Actually, a large number of small-angle collisions are more important in determining $\eta$ than one large-angle collision. These small-angle collisions bring in the density-dependent term $\log\Lambda$.

of plasma in a diffusion-limited regime is proportional to $1/\tau$, rather than varying exponentially with $\tau$.

The results of a variety are, however, consistent with the following empirical Bohm diffusion formula:

$$D_\perp = \frac{1}{16}\frac{k_B T}{eB} \tag{5.19}$$

This "anomalous diffusion" coefficient is four orders of magnitude larger than that given by Eq. (5.17) for the Princeton Model C Stellerator, for example. There are at least these three mechanisms for roughly such anomalous diffusion:

1. Non-alignment:

   The lines of force $\vec{B}$ may leave the chamber and allow the electrons to escape. The ambipolar electric field pulls the ions along, as well.

2. Asymmetric electric field:

   There are $\vec{E}\times\vec{B}$ drifts taking particles into the walls.

3. Fluctuating $\vec{E}$ and $\vec{B}$ fields are induced:

   A "random walk" of plasma particles arises because of fluctuating $\vec{E}\times\vec{B}$ drifts.

One must take care to eliminate these and other sources of anomalous diffusion in order to be able to apply Eq. (5.17). It is vital for controlled fusion to reduce diffusion to that level, and considerable progress is being made on the problem.

## 5.3 Hydromagnetic Equilibrium

We now discuss hydromagnetic equilibrium of a plasma in a magnetic field. The magnetic field induces a current in the plasma because of Ampère's law:

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc} \tag{5.20}$$

This current, in turn, experiences a Lorentz force $i_{enc}\vec{\ell}\times\vec{B}$ or $\vec{j}\times\vec{B}$ on a unit volume of plasma. This force is balanced by the pressure gradient:

$$\text{grad}\, p = \vec{j}\times\vec{B} \tag{5.21}$$

The diamagnetic current force cancels the pressure gradient, and there is no net force on a unit volume. For simplicity we consider an external magnetic field that has an $x$-dependent component. We take an Ampèrian circuit that is a rectangle in the $x-z$ plane to obtain

$$z\,(B_z(x+\Delta x) - B_z(x)) = -\mu_0\, j_y\, z\,(\Delta x)$$

or

$$j_y = -\frac{1}{\mu_0}\frac{dB_z}{dx} \tag{5.22}$$

Consequently

$$\frac{dp}{dx} = j_y B_z = -\frac{1}{\mu_0} B_z \frac{dB_z}{dx}$$

or

$$\frac{d}{dx}\left(p + \frac{B^2}{2\mu_0}\right) = 0 \tag{5.23}$$

The term $B^2/(2\mu_0)$ is called the magnetic field pressure. According to Eq. (5.23), the sum of the "particle pressure" and magnetic pressure is position independent.

In a cylindrical geometry with $\vec{B}$ directed along the axis, current will circulate around the cylinder, and the pressure gradient will be radially inward, as shown:

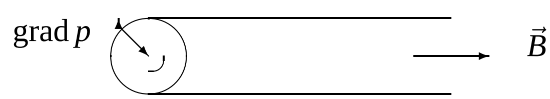


Figure 5.2: Cylindrical Geometry

Therefore, the central region has high particle pressure and low field pressure, whereas in the outer regions the particle pressure is lower and the field (and field pressure) is higher. The ratio of particle pressure to magnetic field pressure is conventionally called β in plasma physics. For reasons of simplicity we have talked mostly about low-β plasmas; say $\beta \approx 10^{-4}$. High-β plasmas are more practical for fusion reactions, however, since the fusion rate is proportional $n^2$ )or $p^2$), whereas the cost increases dramatically (astronomically!) with the magnetic field.

Finally, we shall talk about the diffusion of a magnetic field into a plasma – a question of frequent interest in astrophysical applications.

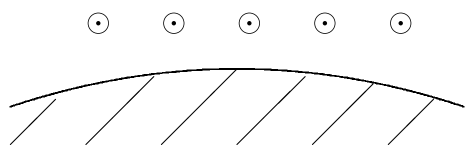


Figure 5.3: Diffusion of Field into Plasma

The surface currents, which are set up in the plasma, exclude the field except over a small surface layer. We can use energy conservation to estimate the “magnetic diffusion time” $\tau$ that is required for the field to penetrate the plasma, by requiring the magnetic field energy to be dissipated by Joule heating. The Joule heating rate per unit volume is $jE = \eta j^2$, so that

$$\tau = \frac{B^2}{2\mu_0 \eta j^2} \tag{5.24}$$

In turn, the current is determined from Ampère’s law, Eq. (5.19), to be of order of magnitude $j = B/(\mu_0 L)$, where $L$ is the length scale over which the magnetic field changes.[3] Consequently

$$\tau \approx \frac{\mu_0 L^2}{\eta} \tag{5.25}$$

For a copper ball of radius 1 cm, $\tau$ is about one second. For the earth’s molten core, $\tau$ is about $10^4$ years. For the magnetic field at the surface of the sun, $\tau \approx 10^{10}$ years.

We mention in passing that $\beta < 1$ is a necessary condition for stability of plasma. Plasmas are subject to various sorts of instabilities, as a result of which initially small disturbances grow with time.

[3]Note that Eq. (5.21) is a particular example consistent with such an estimate.

# Chapter 6

# Kinetic Theory of Plasmas: Nonlinear Effects

## 6.1 Introduction

In previous lectures we have discussed and applied the fluid equations for the components of a plasma. In such applications one makes the fluid approximation. That is to say, we assume that in a small neighborhood of any point $\vec{r}$ in space and at any time $t$, we associate a fliud number density $n(\vec{r},t)$, as well as a fluid velocity $\vec{v}(\vec{r},t)$. We have, in effect, assumed that all the molecules in a given clump of fluid move with a common velocity. However, if we take a small clump of gas at thermal equilibrium, which nevertheless contains many molecules, there is in fact a distribution of molecular velocities within the clump. It is imprecise to associate an "average velocity" to all the molecules, although it is an adequate approximation in many situations. Because of the distribution of molecular velocities, the fluid volume elements break up rather rapidly. The fluid equations, obtained by applying Newton's laws to such fluid volume elements, are thus of questionable validity.

The approach of kinetic theory, which is more general than the fluid approach, is to define $f(\vec{v},\vec{r},t)$ as the probability of finding an atom at a given position $\vec{r}$ and with a specified velocity $\vec{v}$ at time $t$. One can compute the number density $n(\vec{r},t)$ by integrating $f$ over all velocities:

$$n(\vec{r},t) = \int d^3v\, f(\vec{v},\vec{r},t) \tag{6.1}$$

It is conventional to define the distribution $\hat{f}$ by

$$f(\vec{v},\vec{r},t) = n(\vec{r},t)\,\hat{f}(\vec{v},\vec{r},t) \tag{6.2}$$

so that $\hat{f}$ is normalized to unit probability in velocity space:

$$\int d^3v\,\hat{f}(\vec{v},\vec{r},t) = 1 \tag{6.3}$$

A frequently encountered velocity distribution $\hat{f}$ is that of free particles (mass $m$) in thermal equilibrium at temperature $T$:

$$\hat{f}(v) = \left[\frac{m}{2\pi k_B T}\right]^{3/2} \exp\left[-\frac{mv^2}{2k_B T}\right] \tag{6.4}$$

This distribution, called the Maxwellian distribution of velocities, is independent of the position $\vec{r}$, the time $t$, and the direction of the velocity $\vec{v}$. Let us integrate over directions to obtain

$$\int dS_v\,\hat{f}(v) = 4\pi v^2\,\hat{f}(v) = g(v) \tag{6.5}$$

where $g(v)$ is the distribution of speeds, with normalization

$$\int_0^\infty dv\,g(v) = 1 \tag{6.6}$$

The most probable speeds are of order $\sqrt{2k_B T/m}$.

## 6.2 Vlasov Equation

A plasma cannot, in general, be expected to be in thermal equilibrium and to have a Maxwellian distribution of speeds because of the following: drift, loss cone anisotropies, spatial anisostopies, and transverse versus longitudinal temperatures.

Changes in time of the function $f(\vec{r},\vec{v},t)$, which represents the number density in six-dimensional $(\vec{r},\vec{v})$ phase space, are governed by the Boltzmann equation. Let us define the convective time derivative in phase space of the function $f$, as given at a time-dependent position $\vec{r}(t)$ with a time-dependent velocity $\vec{v}(t)$:

$$\frac{Df}{Dt}(\vec{r},\vec{v}(t),t) = \frac{\partial f}{\partial t} + (\mathrm{grad}_{\vec{r}}\,f)\cdot\frac{d\vec{r}}{dt} + (\mathrm{grad}_{\vec{v}}\,f)\cdot\frac{d\vec{v}}{dt} \tag{6.7}$$

We apply Newton's second law to a tiny clump of matter at position $\vec{r}(t)$ moving with velocity $\vec{v}(t)$, so that $d\vec{r}/dt = \vec{v}$ and $d\vec{v}/dt = \vec{F}/m$, where $F$ is the external force acting on the clump. The convective derivative changes only because of collisions with other particles (or clumps) within the plasma, so that

$$Df/Dt = (\partial f/\partial t)(\text{coll})$$

where the character of the collision term on the right is beyond our scope. Thus, the Boltzmann equation is

$$\frac{\partial f}{\partial t} + (\text{grad}_{\vec{r}}\, f) \cdot \frac{d\vec{r}}{dt} + (\text{grad}_{\vec{v}}\, f) \cdot \frac{\vec{F}}{m} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{6.8}$$

The gradient terms on the right of (6.8) take into account the effects of particle drift and external forces. The fluid equations may be derived by integrating the Boltzmann equation over three dimensional velocity space, computing various moments of $f$ with respect to velocity. Boltzmann's equation contains more information than the fluid equations, so that one cannot derive the Boltzmann equations from the fluid equations. Conversely, some version of the fluid equations is correct, but one cannot be sure about the simplifying assumptions to analyze it.

It is a reasonable approximation to neglect collisions altogether in a hot plasma. If we do so, and include only electromagnetic forces, we obtain the Vlasov equation, which pertains more directly to plasma physics:

$$\frac{Df}{Dt}(\vec{r}(t), \vec{v}(t), t) = \frac{\partial f}{\partial t} + (\text{grad}_{\vec{r}}\, f) \cdot \frac{d\vec{r}}{dt} + \frac{q}{m}\,(\text{grad}_{\vec{v}}\, f) \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) = 0 \tag{6.9}$$

Let us discuss electron plasma oscillations in the context of the Vlasov equation. We will obtain a dispersion relation $\omega(k)$ which reduces in an appropriate limit to the electron plasma wave formula

$$\omega^2 = \omega_P^2 + (3k_B T/m)\, k^2$$

The dispersion formula gives $\omega$ a small imaginary part, to indicate that electron oscillations are damped in a plasma. This damping, called Landau damping, was predicted as a result of a subtle theoretical analysis before it was actually seen in laboratory experiments.

Let us consider a uniform plasma, with no external electric or magnetic fields. We shall assume for simplicity that the unperturbed Boltzmann distribution is the

Maxwellian distribution of velocities, $f_0(v)$. The electron plasma wave adds a small disturbance to this distribution:

$$f(\vec{r},\vec{v},t) = f_0(v) + f_1(\vec{r},\vec{v},t) \tag{6.10}$$

where $f_1$ is a sinusoidally propagating disturbance of wave number $k$ and frequency $\omega$:

$$f_1(\vec{r},\vec{v},t) = f_1 e^{i(kx-\omega t)} \tag{6.11}$$

with $f_1$ independent of position and times. A weak electric field is induced by the instantanteous charge imbalance in the plasma:

$$\vec{E}_1(\vec{r},t) = \hat{i} E_1 e^{i(kx-\omega t)} \tag{6.12}$$

We keep only the first order terms in $f_1$ and $E_1$ in the electron Vlasov equation to obtain

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \text{grad}_{\vec{r}}\, f_1 - \frac{e\vec{E}_1}{m} \text{grad}_{\vec{r}}\, f_1 = 0 \tag{6.13}$$

The factor $\exp[i(kx-\omega t)]$ drops out of (6.13), and we obtain for plasma oscillations in the $x$-direction

$$-i\omega f_1 + v_0 (ik f_1) - \frac{eE_1}{m} \left( \frac{\partial f_0}{\partial v_0} \right) = 0$$

or

$$f_1 = i \frac{eE_1}{m} \frac{1}{\omega - ck_0} \left( \frac{\partial f_0}{\partial v_0} \right) \tag{6.14}$$

We must also use Gauss's law of electrostatics to find an additional relation between the nonequilibrium distribution of electrons $f_1$ and the induced electric field $\vec{E}_1$. Let us take a cylindrical Gaussian pillbox of cross-sectional area $A$ with end surfaces at $x = 0$ and $x = x$. We obtain

$$\varepsilon_0 \oint \vec{E} \cdot \vec{dS} = \varepsilon_0 A (E(x) - E(0)) = q_{enc} = \int_0^x dy\, (-en_1(y))A \tag{6.15}$$

Differentiate with respect to $x$ to obtain

$$\varepsilon_0 \frac{dE}{dx} = -en_1(x) = -e \int d^3v f_1(\vec{r},\vec{v},t) \tag{6.16}$$

We have made the assumption that the unperturbed plasma is electrically neutral. We insert the plane-wave space-time dependence into (6.16) and use (6.14) to obtain

$$ik\varepsilon_0 E_1 = -e \int d^3v\, f_1(\vec{v})k = -i \frac{e^2 E_1}{m} \int \frac{d^3v}{\omega - kv_x} \left( \frac{\partial f_0}{\partial v_x} \right) \tag{6.17}$$

The electic field amplitude $E_1$ cancels out, leaving the dispersion formula

$$-\frac{e^2}{m k \varepsilon_0} \int \frac{d^3 v}{\omega - k v_x} \left(\frac{\partial f_0}{\partial v_0}\right) = 1 \tag{6.18}$$

We factor out the number density $n_0$ from the Maxwellian distribution ($f_0 = n \hat{f}_0$) to obtain

$$-\frac{\omega_P^2}{k} int \int dv_y dv_z \int_{-\infty}^{\infty} \frac{dv_x}{\omega - k v_x} \left(\frac{\partial \hat{f}_0}{\partial v_0}\right) = 1 \tag{6.19}$$

we perform the integrations over $v_y$ and $v_z$ to obtain

$$\frac{\omega_P^2}{k^2} \int_{-\infty}^{\infty} \frac{dv_x}{v_x - \omega/k} \left(\frac{\partial \hat{g}_0}{\partial v_0}\right) = 1 \tag{6.20}$$

where $\hat{g}_0(v_0)$ is the one-dimensional Maxwellian distribution

$$\hat{g}_0(v_0) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} e^{-m v^2/(2 k_B T)} \tag{6.21}$$

## 6.3 Landau Damping

Eq. (6.20) gives the angular frequency ω as an implicit function of the wave number $k$. Such an implicit relation may be inconvenient for certain purposes; but, for example ω may be determined numerically from $k$ quite readily. A more serious problem is that the integration over $v_0$ in (6.20) diverges because the demoninator vanishes at $v_x = \omega/k$. This divergence arises because, in fact, plane waves of this type cannot propagate without attenuation (or growth, possibly) in the plasma. One may analyze this matter properly by taking the deviation from the equilibrium distribution, $f_1$, at $t = 0$. One needs to determine whether this distribution grows or decays with time. The angular frequency becomes a complex function of the wave number $k$. The time-dependent factor is

$$e^{-i\omega t} = e^{-i\omega_R t} e^{\omega_I t} \tag{6.22}$$

If the imaginary part of ω is negative one has attenuation, whereas if $\omega_I$ is positive one has exponential growth. The Russian physicist Lev Davidovich Landau showed that, as a result of requirements of causality on a localized wave packet, one should treat the integral (6.20) as a contour integral in the complex $v_x$ plane. In particular, the contour integral should be placed an infinitesimal distance under the real $v_x$ axis, and under the pole at $v_x = \omega/k$, as shown.

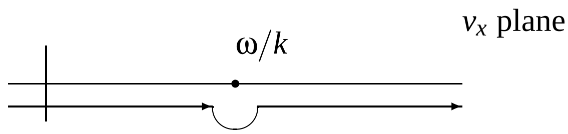


Figure 6.1: Integration contour

According to the standard lore of complex variable integration, one may write that integral as

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{dv_x}{v_x - \omega/k} \left(\frac{\partial \hat{g}_0}{\partial v_x}\right) + i\pi \frac{dv_x}{v_x - \omega/k} \left(\frac{\partial \hat{g}_0}{\partial v_0}(v = \omega/k)\right) \tag{6.23}$$

The first term in (6.23) is the Cauchy principal value integral, defined as

$$\lim_{\delta \to 0^+} \left[\int_{-\infty}^{\omega/k-\delta} + \int_{\omega/k-\delta}^{\infty}\right] \frac{dv_x}{v_x - \omega/k} \left(\frac{\partial \hat{g}_0}{\partial v_0}\right) \tag{6.24}$$

The Cauchy Principal value corresponds to integration along the straight sections of the $v_x$ integration contour, whereas the second term in (6.23) comes from the semi-circular arc centered at $v_x = \omega/k$.

We shall analyze the above dispersion relation in the physically realistic case, in which the velocity $v_x = \omega/k$ is far out on the tail of the Maxwellian distribution $\hat{g}_0(v)$. In such a case, the principal value integral is dominated by contributions at small $v_x$; i. e. $v_x \ll \omega/k$. Let us integrate the principal value integral by parts:

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{dv_x}{v_x - \omega/k} \left(\frac{\partial \hat{g}_0}{\partial v_x}\right) = \frac{g_0}{v_x - \omega/k}\bigg|_{-\infty}^{\infty} + \mathrm{P}\int_{-\infty}^{\infty} \frac{dv_x}{(v_x - \omega/k)^2} g_0(v_x) \tag{6.25}$$

The integrated term vanishes, and the second term is dominated by small $v_x$. We expand the denominator in the latter in powers of $v_x$ to obtain

$$\begin{aligned} &\frac{k^2}{\omega^2} \mathrm{P}\int_{-\infty}^{\infty} dv_x\, g_0(v_x) \left[1 - \frac{v_x k}{\omega}\right]^{-2} \\ \approx\ &\frac{k^2}{\omega^2} \mathrm{P}\int_{-\infty}^{\infty} dv_x\, g_0(v_x) \left[1 + 2\frac{v_x k}{\omega} + 3\left(\frac{v_x k}{\omega}\right)^2 + \cdots\right] \end{aligned} \tag{6.26}$$

The integrals over $v_x$ are moments of the Maxwellian distribution, and one gets

$$\frac{k^2}{\omega^2} + 2\frac{k^3}{\omega^3} < v_x > + 3\frac{k^4}{\omega^4} < v_x^2 > + \cdots = \frac{k^2}{\omega^2}\left[1 + \frac{k^2}{\omega^2}\frac{3k_B T}{m}\right] \tag{6.27}$$

In this regime of approximation, the dispersion relation (6.20) becomes

$$1 = \frac{\omega_P^2}{\omega^2}\left[1 + \frac{3k_B T}{m}\frac{k^2}{\omega^2}\right] + i\pi\frac{\omega_P^2}{\omega^2}\left[\frac{\partial \hat{g}_0}{\partial v_0}\right](\omega/k) \tag{6.28}$$

The solution to this equation is close to $\omega = \omega_P$, since the last two terms in (6.28) are small in the region of interest. Thus,

$$\omega^2 = \omega_P^2\left[1 + \frac{3k_B T}{m}\frac{k^2}{\omega^2} + \frac{i\pi\omega_P^2}{k^2}\left(\frac{\partial \hat{g}_0}{\partial v_0}\right)\right] \tag{6.29}$$

or

$$\omega = \omega_P + \frac{3k_B T}{m}\frac{k^2}{\omega_P} + \frac{i\pi\omega_P^2}{2k^2}\left[\frac{\partial \hat{g}_0}{\partial v_0}\right]\left(\frac{\omega}{k}\right) \tag{6.30}$$

Since on the tail of the Maxwellian distribution $\hat{g}_0(v)$ descreases as $v$ increases, the third term in $\omega$ gives a small negative imaginary part. Consequently, the electron plasma wave is damped as it propagates through plasma. The energy in the wave is converted into energy of particle motion as a result of coupling through the electric field.

We obtain a physical picture of Landau damping by observing that electrons can potentially "ride between the crests of the electric potential"; i. e., they like to "lock in" with their velocity equal to the phase velocity of the wave $v_P = \omega/k$. (like surfers) The particles with velocities slightly greater than $v_P$ decelerate, and the wave gains energy from them. Conversely, particles with velocities slightly less than $v_P$ are accelerated, and the wave loses energy from them. Since there are more electrons in the latter category the wave experiences a net loss of energy and is attenuated. We mention in passing that Landau damping of ion acoustic waves also occurs.

We have confined the discussion thus far to linear Landau damping, for which the electron plasma wave is exponentially attenuated. Large amplitude plasma waves are seen to be attenuated; then to grow, and to continue to increase in amplitude before settling down to a steady-state value. At that steady-state value, the amplitude of the potential energy $e\phi$ is roughly equal to the kinetic energy of particles moving with the phase velocity of the wave.

$$e\phi_1 \approx m\left(\frac{\omega}{k}\right)^2 \tag{6.31}$$

or

$$\omega \approx \omega_B = \left[\frac{e k^2 \phi_1}{m}\right]^{1/2} \tag{6.32}$$

The frequency $\omega_B$ is roughly the "bounce" frequency of a particle that is trapped in the electromagnetic potential

$$\phi(x) = -\phi_1 (1 - \cos kx) \approx -\tfrac{1}{2}\phi_1 k^2 x^2 \tag{6.33}$$

The equation of motion of small oscillations about the point of minimum potential is

$$m\frac{d^2x}{dt^2} = -eE = e\frac{d\phi}{dx} = -ek^2\phi_1 x \tag{6.34}$$

so that

$$\omega_B^2 = ek^2\phi_1/m$$

For $\omega > \omega_B$ in plasma, one is in a regime of nonlinear Landau damping. An example of a nonlinear effect is the modulation of two high-frequency electron waves. It may be possible to trap particles (ions) in the electric potential corresponding to the envelope of the modulated wave, so that the particles move with exactly the group velocity of the wave packet.

## 6.4 Plasma Sheath

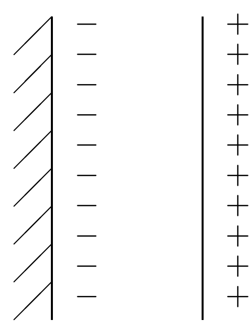

Figure 6.2: Sheath Region

We will discuss another nonlinear effect – the existence of a sheath in a plasma. For a plasma contained within a finite volume, one expects the electrons and ions to recombine at the walls of the plasma. At least for the simple case in which there is no magnetic field, one expects the scalar potential $\phi$ to be zero in the bulk of the plasma. However, the electrons are more mobile than the ions, so that, in a steady-state condition there is always a net negative charge on the walls and a net positive charge in the plasma near the walls. (An electric field is set up near the walls that accelerates ions and decelerates ions, causing them to arrive at the wall at the same rate.)

The electric potential $\phi(x)$ is negative in a sheath region of thickness of order a Debye length. If the ions have drift velocity $u_0$ at the edge of the sheath (where $\phi = 0$), then conservation of energy requires

$$\tfrac{1}{2} m u_0^2 = \tfrac{1}{2} m u^2 + e\phi(x) \tag{6.35}$$

where $u$ is the velocity of the ions at position $x$. One must also use the ion equation of continuity, the Boltzmann equation to determine the electron density, and Gauss's law of electrostatics. One finds the following requirement for the existence of a sheath, the Bohm sheath criterion.

$$u_0 \geq \sqrt{\frac{k_B T}{m}} \approx v_s \tag{6.36}$$

In other words, the ion must enter the plasma at a velocity greater than the velocity of sound. (We have considered a cold ion plasma, $T_i \ll T_e$, for convenience.)

Finally, we mention the intrinsically nonlinear phenomenon of the ion acoustic shock wave. This shock is set up whenever a plasma is drifting around an object with a speed greater than that of sound – in effect a sheath is set up around the object. The shock wave is very similar to the sonic boom set up by a jet in the atmosphere; for $v > v_s$ there are no precursors and a large amplitude shock wave builds up. A similar effect is the "bow shock" of the earth as it drags its magnetic field through the surrounding plasma, the solar wind.

# Chapter 7

# Controlled Fusion: Confining and Heating Plasma

## 7.1 Nuclear Fusion

This final lecture deals with the challenging and crucial problem of obtaining controlled nuclear fusion in the laboratory, and using it as a virtually limitless source of energy. All serious schemes for accomplishing this goal involve plasmas, since one must get matter to high temperatures before nuclear reactions can occur. We shall first discuss the physics of fusion reactions rather generally, and then talk about the promising reactions for practical energy production.

Let us consider the fusion reaction

$$a + b \rightarrow \text{fusion products} + W \tag{7.1}$$

where $a$ and $b$ are the initial nuclei, and $W$ is the kinetic energy released by the fusion reaction – typically $W$ is a few MeV. This fusion reaction cannot occur, however, unless nuclei can overlap, or get with a distance of about $10^{-13}$ cm from one another. In particular, they must overcome the Coulombic repulsion by having sufficient kinetic energy when they are separated.

$$Z_a Z_b \frac{e^2}{4\pi\varepsilon_0 d} = 10^4 Z_a Z_b \qquad \text{(eV)} \tag{7.2}$$

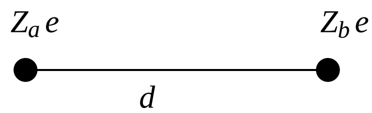

Figure 7.1: Coulombic Repulsion of Nuclei

There may be additional factors (such as centrifugal barrier effects) to require a somewhat larger energy for a particular fusion reaction to occur with substantial probability. As the temperature of the plasma is increased, the particles have more kinetic energy, and the fusion reactions can begin to occur at the ignition temperature. The power produced by the fusion reaction per unit volume is

$$P_{fusion} = n_a n_b W \tag{7.3}$$

where $n_a$ and $n_b$ are the number density of species $a$ and $b$, respectively. The reaction cross-section $\sigma$ is of order $\pi d^2$, or $10^{-26} cm^2$, if the nuclei have enough energy to get close. The quantity $v$ is the collision speed. One must average $\sigma v$ over the actual speed distribution in the plasma: note that $\sigma$ is a rapidly varying function of the speed.

The fusion energy is not the whole story, however, since the plasma has natural mechanisms for “cooling off”, or radiating energy. Recall that a black body at temperature $T$ radiates energy at a rate proportional to $T^4$. Fortunately, a plasma does not lose energy as rapidly as a black body – if it did one would radiate away more energy than it could possibly produce in the laboratory by fusion. The dominant mechanism for radiation of energy in a plasma is Bremsstrahlung, or braking radiation. If $n$ is the number density of ions,, $Z$ is their average charge, and $T$ is the ion temperature, then the power radiated is proportional to

$$Z^2 n^2 \sqrt{k_B T}$$

It happens through a lucky circumstance that one can produce more power by fusion than is radiated away by Bremsstrahlungabove the ignition temperature, under favorable conditions.

An additional loss of energy comes from the escape of particles. At best one simply loses energy; at worst one can damage the walls of the container, destroy the supoerconducting magnets, contaminate the plasma with impurities, and release harmful radiation. (These harmful catastrophes can be avoided, according to experts.)

The Lawson criterion is a condition necessary to produce more energy by fusion than is lost by Bremsstrahlung and escape, if the fusion energy is recovered at $1/3$ efficiency. The condition is on the product of the plasma number density $n$ (number per cubic centimeter) and the time $\tau$ (sec) for which the plasma is confined above the ignition temperature. For $DT$ reactions, the most favorable case, the Lawson criterion is

$$n\tau \geq 10^{14} \tag{7.4}$$

For other fusion reactions it is much higher.

In the usual plasma confinement schemes one hopes to get a density $n = 10^{15}$ per cubic centimeter ane confinement times $\tau = 0.1$ second to achieve "break-even" conditions. By imploding a pulsed laser beam on a solid hydrogen pellet, one tries to get hot plasma with $n = 10^{26}$ particles per cubic centimeter for confinement times $\tau \approx 10^{-11}$ seconds to achieve laser driven fusion. There are also projects for accelerator fusion, in which an intense heavy ion beam is used to heat the plasma. These latter two schemes have much to offer, but in these lectures we will discuss only the more conventional schemes for confining plasma and heating it.

The most commonly considered fusion raction is the deuterium-tritium reaction

$$D + T \rightarrow He^4\,(3.5\text{ MeV}) + n \quad (14.4\,\text{MeV}) \tag{7.5}$$

The ingredients, $D$ and $Li^6$, are readily available from natural sources. The $D-T$ reaction has the lowest ignition temperature of any fusion reaction, 4 KeV. Moreover, most of the fusion energy is released in the form of 14 MeV neutrons. It gets converted into thermal energy, where it can only rather inefficiently be recovered. The kinetic energy of charged fusion products may be directly converted into electrical energy. Finally, the energetic neutrons may cause radiation damage for this $D-T$ reaction to materials in the confinement system. The Lawson number for this $D-T$ reaction is $10^{14}$.

Another reaction commonly considered is the $D-D$ reaction. There are two sets of reaction products, which occur with equal probability:

$$\begin{aligned} D + D &\rightarrow T\ (1\,\text{MeV}) + p\ (3\,\text{MeV}) \\ &\rightarrow He^3\ (.8\,\text{MeV}) + n\ (2.5\,\text{MeV}) \end{aligned} \tag{7.6}$$

This case has the advantage that tritium breeding is not necessary, and only one-third of the fusion energy is given to neutrons. Its disadvantages are a higher ignition temperature (35 KeV) and Lawson number ($10^{16}$).

We lits three other reacitons which mave the desirable feature of producing no neutrons at all, but which require still higher ignition temperatures and Lawson numbers for fusion:

$$
\begin{aligned}
D + He^3 &\rightarrow He^4 + p \quad (18.3\,\text{MeV}) \\
p + Li^6 &\rightarrow He^4 + He^3 \quad (4.0\,\text{MeV}) \\
p + B^{11} &\rightarrow 3He^4 \quad (8.7\,\text{MeV})
\end{aligned}
\tag{7.7}
$$

## 7.2 Plasma Pinching

The three most common plasma heating and confinement schemes involve magnetic mirrors (discussed previously), theta pinches, and toruses. We shall discuss the "pinching" of plasma in some detail, before briefly describing toruses.

In order to produce a pinch in plasma, one passes a current through the plasma. The current heats the plasma and produces a magnetic field $\vec{B}$, and the Lorentz force

$$\vec{F} = q\vec{v} \times \vec{B}$$

pinches the plasma. As one increases the current. the plasma becomes more confined, and its temperature increases. We shall describe both the $z$-pinch and the $\theta$-pinch for a cylindrical plasma.

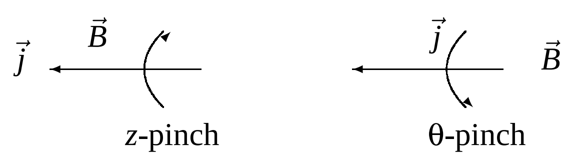


Figure 7.2: Pinches of Cylindrical Plasma

In the $z$-pinch current flows in the direction of the axis of the cylinder, and the field has only a $\theta$-component. By contrast, for the $\theta$-pinch, the magnetic field lies along the $z$-axis, whereas the current circulates in the $\theta$-direction.

We shall consider the $z$-pinch case in some detail. An electric field $\vec{E}$ lies in the $z$-direction, inducing a current $\vec{j}$ also in that direction. (Note that electrons and ions each produce such a current.) This current, in turn, induces a magnetic field $\vec{B}$ in the $\theta$-direction, so that the Lorentz force ($\vec{F} = q\vec{v} \times \vec{B}$) lies radially inward. This force induces a self-constriction of the plasma, which is opposed by the kinetic pressure of the plasma. The equation of motion on a unit volume of the plasma is

$$nM\frac{D\vec{v}}{Dt} = \vec{j}\times\vec{B} + \mathrm{grad}\, p \tag{7.8}$$

where for quasi-equilibrium the left side of this equation is zero, and where

$$n_i = n_e = n(\vec{r}\,)$$

The current is dominated by the lighter and more mobile elections:

$$\vec{j} = ne(\vec{v}_i - \vec{v}_e) = -ne\vec{v}_e \tag{7.9}$$

The pressure in the plasma is

$$p = nk_B(T_i + T_e) \tag{7.10}$$

so that

$$ne\vec{v}_e\times\vec{B} + k_B(T_i+T_e)\,\mathrm{grad}\, p = 0 \tag{7.11}$$

The electron velocity has only a $z$-component, $\vec{B}$ has only a $\theta$-component, and the density gradient is radial, so we obtain

$$nev_eB_\theta + k_B\,(T_e+T_i)\,\frac{dn}{dr} = 0 \tag{7.12}$$

We must supplement the fluid equation with Ampère's law, which relates the current to the field.

$$\oint \vec{B}\cdot\vec{d\ell} = \mu_0\, i_{enc} = \mu_0\int \vec{j}\cdot\vec{dS} \tag{7.13}$$

For a ring of radius $r$ centered on the cylinder axis we obtain

$$B_\theta(2\pi r) = \mu_0\int_0^r 2\pi r'\, j(r')\,dr' = -\mu_0\int_0^r 2\pi r'\, nev_e \tag{7.14}$$

or

$$\frac{1}{r}\frac{d}{dr}(rB_\theta) = -\mu_0\, nev_e \tag{7.15}$$

One may solve the coupled system of equations (7.12) and (7.15), to obtain the Bennett distribution of particles:

$$n(r) = \frac{n_0}{(1+n_0br^2)^2} \tag{7.16}$$

where

$$b = \frac{\mu_0 e^2 v_e^2}{8k_B(T_i + T_e)} \tag{7.17}$$

and $n_0$ is the plasma density at $r = 0$. (We have assumed that $v_e$ is independent of $r$.) Most of the plasma lies within the Bennett radius,

$$r_e = [n_0 b]^{-1/2} \tag{7.18}$$

The number of particles per unit cylinder length is

$$N_\ell = \int_0^\infty dr\, 2\pi r\, n(r) = \frac{\pi}{b} \tag{7.19}$$

The total current flowing in the plasma is

$$I = N_\ell e v_e \tag{7.20}$$

Let us use (7.19) and (7.20) to eliminate $b$ and $v_e$ from (7.17) to obtain the Bennett relation between the current $I$, the particle number $N_\ell$, and the temperatures:

$$I^2 = \frac{8\pi N_\ell}{\mu_0} k_B(T_e + T_i) \tag{7.21}$$

This relation happens to be correct even if the electron velocity $v_e$ depends on $r$. By passing the proper current $I$ through a plasma, one can heat the plasma to any temperature – in practice the temperature is limited by Bremsstrahlung losses, of course. As one increases the current (and thus the temperature) the Bennett radius decreases – the plasma is pinched.

Because of certain instabilities that are produced in the plasma pinch, which modify the the pinch after a time, it is more practical to consider the dynamic pinch of plasma, rather than the more direct quasi-equilibrium pinch. In dynamic pinch, one discharges a bank of capacitors, producing a large transient discharge current in the plasma. Actually, current flows only within a Debye length from the surface of the plasma, because of Debye shielding. The inward Lorentz force pulls all particles near the surface of the plasma inward; a “snow plow effect” is thus built up. The plasma continues to collapse until the point at which kinetic pressure balances magnetic pressure, yielding a Bennett distribution.

The cylindrically symmetric pinch of plasma is unstable, however, At the surface of the plasma, the kinetic pressure balances the magnetic pressure. However, if there is a small bulge in the plasma surface, the magnetic pressure is a little less than on the unperturbed surface. Consequently, the kinetic pressure exceeds the magnetic pressure in the bulge, so that the bulge expands with time. Two common types of instabilities, the “kink” and “sausage” instabilities, are shown:

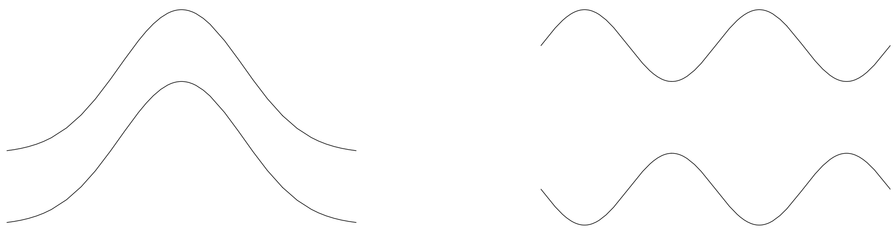


Figure 7.3: Kink and Sausage Instabilities

These instabilities are somewhat less severe with the $\theta$-pinch than with the $z$-pinch, so that in practice the former is more commonly used.

## 7.3 Toroidal Devices

Finally, let us discuss toroidal devices. The simple toroidal configuration (toroidal windings with no pitch) is inappropriate. If one has $N$ windings with current $I$ flowing, it follows from Ampère's law, Eq. (7.13), that the magnetic field is axially symmetric, with

$$B_\theta = \frac{\mu_0 N I}{2\pi r} \tag{7.22}$$

As we have discussed previously, there are both grad $B$ and curvature drifts, which are in the same direction. These drifts give rise to a net separation of charge: more positive charges on the top of the container; more negative charges on the bottom. The charge separation produces an electric field to cancel out these drifts. However, there must then be $\vec{E} \times \vec{B}$ drifts, with velocity

$$\vec{v} = \frac{\vec{E} \times \vec{B}}{B^2} \tag{7.23}$$

which produce radially outward motion of ions and electrons.

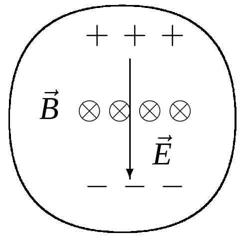


Figure 7.4: Toroidal Drifts

One may eliminate the problem by putting a twist in the lines of force of the magnetic field; the field lines must wrap around the torus.

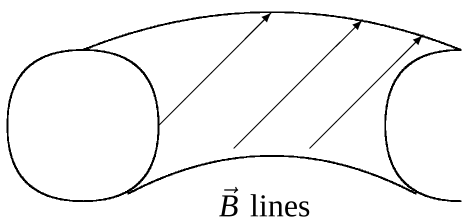


Figure 7.5: Toroidal Drifts

When the twist is present, the grad$B$ and curvature drifts are averaged out as the particles go around the torus. However, this twisted magnetic field introduces certain instabilities in the plasma.

- Because the field lines are curved, there is a gravitational (inertial) instability.
- A two-stream electromagnetic instability may occur, since the electrons and ions can have different drift speeds.
- There is an electromagnetic instability, which produces kinks in the toroidal configuration.

These instabilities may be kept under control by inducing magnetic shear. That is, the magnetic field along a concentric torus with the same major radius and smaller minor radius must have a greater twist than for the corresponding larger minor radius.

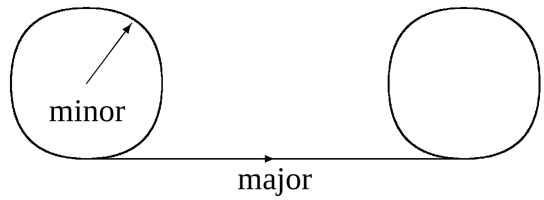


Figure 7.6: Major and Minor Radius

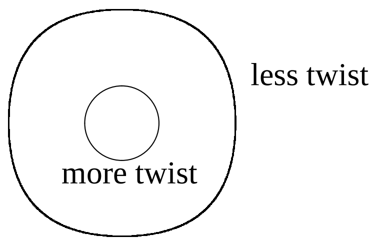


Figure 7.7: Twist for Various Minor Radii

We shall discuss the means by which twist and shear are produced in stellerators and tokomaks,[1] the two main types of toroidal devices.

The stellerator has helical field windings to produce twist and shear in the magnetic field lines. Unfortunately, not enough magnetic shear is produced by such means to bring the instabilities under control. Also, the magnetic field is not axially symmetric, so that the machine is difficult to align, and any asymmetric electric fields can produce drifts in the plasma.

The most promosing toroidal device is the tokomak, which was proposed by the Russian plasma physicist Artsimovich. The external windings produce the usual simple toroidal field – this “external” magnetic field is axially symmetric, and hence easier to build and analyze. An internal plasma current produces an “internal” magnetic field, so that the total magnetic field lines have the proper amount of twist and shear. Finally, a vertical magnetic field prevents the internal current ring from expanding – the Lorentz force generated by it cancels out the magnetic pressure on the ring. It has been found that short, far tokomaks of non-circular cross section are most suitable for fusion. Plasma physicists fully expect to achieve “break-even” conditions on fusion in the near future.

[1]For a discussion of the history of Tokomaks, as well as the meaning of the word as a Russian language acronym, see the following website: http://en.wikipedia.org/wiki/Tokamak.

# Appendix A

# Examination

Part I

1. What is a plasma? Give as many different examples as you can.
2. Explain why plasmas necessarily occur either at high temperatures (above $10^5$ K) or with very low densities (below $10^8$ particles per cubic centimeter).
3. Briefly give as many applications or potential applications as you can.
4. What is Debye shielding of an electric field inside a plasma? Why does the Debye length increase with temperature and decrease with density?
5. A particle with charge $q$ in a uniform external magnetic field $\vec{B}$, when subject to an additional constant force $\vec{F}$, experiences a guiding center drift at velocity
$$\vec{v}_D = \vec{F} \times \vec{B}/(qB^2)$$
Explain qualitatively why the drifts arise?
6. What is a magnetic mirror, and how is it used to confine plasma? What is a "loss cone"?
7. Describe the motion of charged particles that are trapped in Van Allen belts by the earth's magnetic field.
8. Why are plasmas diamagnetic media?

9. A fluid plasma of $n$ particles per unit volume of charge $q$ is placed in a uniform $\vec{B}$ field. In response to a pressure gradient grad $p$, a plasma experiences a diamagnetic drift of velocity

$$\vec{v}_D = -(\text{grad}\, p \times \vec{B})/(q\,n\,B^2)$$

Give a qualitative physical explanation of this drift.

10. A plasma with $n$ particles (of mass $m$ and charge $e$) per unit volume undergoes oscillations with the plasma frequency

$$\omega_P = \sqrt{n\,e^2/(m\varepsilon_0)}$$

where $\varepsilon_0$ is the constant in Gauss's law. What mechanism sustains these oscillations, and why does $\omega_P$ increase with $n$?

11. Describe ambipolar diffusion of electrons and ions in a plasma.

12. Why does a uniform magnetic field reduce diffusion in the direction perpendicular to the field?

13. An electromagnetic wave passing through plasma with no external magnetic field has space-time dependence

$$\exp[i(kx - \omega t)]$$

where the angular frequency $\omega$ is given in terms of the wave number $k$ as

$$\omega^2 = \omega_P^2 + c^2\,k^2$$

Why are waves of frequency $\omega < \omega_P$ reflected from such a plasma?

14. Why does the electrical resistivity of a plasma descrease with temperature? What is "electron runaway" in a high temperature plasma?

15. The time required for a magnetic field to diffuse into a plasma is of order

$$L^2/(\mu_0\,\eta)$$

where $\eta$ is the plasma resistivity, $L$ is the plasma radius, and $\mu_0$ is the constant in Ampère's Law. (For the sun, the diffusion time is $10^{10}$ years.) What physical mechanisms determine the diffusion time, and how?

16. What is “Landau damping” of an electromagnetic wave in plasma? How and under what circumstances does it occur?

17. Discuss the relative merits of these three fusion reactions:

$$
\begin{aligned}
D+T &\rightarrow He^4+n \qquad (17.6\,\mathrm{MeV}) \\
D+D &\rightarrow T+p \qquad (4\,\mathrm{MeV}) \\
D+He^3 &\rightarrow He^4+p \qquad (18.3\,\mathrm{MeV})
\end{aligned}
$$

18. Explain how and why a plasma is pinched when an electric current is passed though it.

19. What energy loss mechanisms are most relevant for heating and confining plasma so as to produce energy by fusion? What is the “Lawson criterion” for break-even energy production? Why must a plasma be heated to $10^8$ K before fusion reactions occur?

20. Simple toroidal plasma are subject to both grad $B$ and curvature drifts. How are these drifts dealt with in Stellerators and in Tokomaks?

## Part II

Define or explain as many of these as you can:

| | |
|---|---|
| Adiabatic drift of plasma | Magnetic twist and shear |
| Alfvén waves | Magnetic pressure; $\beta$ |
| Bennett relation | Maxwellian distribution |
| Bohm (anomalous) diffusion | Phase and group velocity |
| Boltzmann equation | Plasma re-entry blackout |
| Convective derivative | Plasma sheath |
| Faraday rotation | Polarization drift |
| Ion acoustic waves | Saha equation |
| Gravitational instability | Sausage/kink instability |
| Larmor radius | Whistlers |

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